# Lambda Calculus

## Syntax

```
i  ::=  a | b | c | d | e | ..... etc
L  ::=  i
     |  i → L
     |  L L
     |  ( L )
```

## A Few Examples

x
x y
a→x
a→(b a)
c→d→(c e→(d e))
(x→(x x)) (y→(y a))

## Semantics With Examples

α Anywhere that something of the form x→Y appears, the name x may be uniformly replaced by any other name that does not already appear in Y.

| | | |
|---|---|---|
| a→(b a) | ≡ | x→(b x) |
| x→x | ≡ | y→y |
| x→y | ≡ | a→y |
| (a (b→(a b))) | ≡ | (a (p→(a p))) |
| a→(a (b→(a b))) | ≡ | x→(x (b→(x b))) |
| x→(x (b→(x b))) | ≡ | x→(x (y→(x y))) |

β Anywhere that something of the form (x→Y Z) appears, so long as there is no possible confusion between the names appearing in Y and Z, it may be replaced by a copy of Y in which every occurrence of x has been replaced by a copy of Z.

| | | |
|---|---|---|
| (x→(x a)) b | ≡ | (b a) |
| (x→(y a)) b | ≡ | (y a) |
| (x→(x a)) (a b) | ≡ | ((a b) a) |
| (x→x) (y→y) | ≡ | y→y |
| (x→(x x)) (a b) | ≡ | ((a b) (a b)) |
| a→((b→(c b)) e) | ≡ | a→(c e) |

A Larger Example

$$
\begin{aligned}
&(x\to(x\ (x\ y)))\ (a\to(a\ a))\\
\equiv\ &((a\to(a\ a))\ (a\to(a\ a))\ y)\\
\equiv\ &((a\to(a\ a))\ y)\ (a\to(a\ a))\ y))\\
\equiv\ &((y\ y)\ (a\to(a\ a))\ y))\\
\equiv\ &((y\ y)\ (y\ y))
\end{aligned}
$$

Here is a sequence of values following a pattern:

| | | |
|---|---|---|
| $N_0$ | is | $f\to x\to x$ |
| $N_1$ | is | $f\to x\to(f\ x)$ |
| $N_2$ | is | $f\to x\to(f\ (f\ x))$ |
| $N_3$ | is | $f\to x\to(f\ (f\ (f\ x)))$ |
| $N_4$ | is | $f\to x\to(f\ (f\ (f\ (f\ x))))$ |
| $N_5$ | is | $f\to x\to(f\ (f\ (f\ (f\ (f\ x)))))$ |
| | *etc.* | |

And something to go with them

| | | |
|---|---|---|
| I | is | $n\to g\to y\to((n\ g)\ (g\ y))$ |

What is (I $N_0$)?

(I $N_0$) $\equiv$ $(n\to g\to y\to((n\ g)(g\ y)))\ (f\to x\to x)$

---

$(n\to g\to y\to((n\ g)(g\ y)))\ (f\to x\to x)$

$\equiv$ $g\to y\to((f\to x\to x)\ g)\ (g\ y))$

---

$g\to y\to(((f\to x\to x)\ g)\ (g\ y))$

$\equiv$ $g\to y\to((x\to x)\ (g\ y))$

---

$g\to y\to((x\to x)\ (g\ y))$

$\equiv$ $g\to y\to(g\ y)$

---

$g\to y\to(g\ y)$

$\equiv$ $g\to x\to(g\ x)$

$\equiv$ $f\to x\to(f\ x)$

$\equiv$ $N_1$

What is (I $N_4$)?

(I $N_4$) ≡ (n→g→y→((n g)(g y))) (f→x→(f (f (f (f x)))))

---

(n→g→y→((n g)(g y))) (f→x→(f (f (f (f x)))))
≡ g→y→(((f→x→(f (f (f (f x))))) g) (g y))

---

g→y→(((f→x→(f (f (f (f x))))) g) (g y))
≡ g→y→((x→(g (g (g (g x))))) (g y))

---

g→y→((x→(g (g (g (g x))))) (g y))
≡ g→y→(g (g (g (g (g y)))))

---

g→y→(g (g (g (g (g y)))))
≡ g→x→(g (g (g (g (g x)))))
≡ f→x→(f (f (f (f (f x)))))
≡ $N_5$

Something new

A is z→(z I) which really should be written as
z→(z (n→g→y→((n g) (g y))))

What is ((A $N_3$) $N_4$)?

≡ ((z→(z I) (f→x→(f (f (f x))))) $N_4$)

---

((z→(z I) (f→x→(f (f (f x))))) $N_4$)
≡ (((f→x→(f (f (f x)))) I) $N_4$)

---

(((f→x→(f (f (f x)))) I) $N_4$)
≡ ((x→(I (I (I x)))) $N_4$)

---

((x→(I (I (I x)))) $N_4$)
≡ (I (I (I $N_4$)))

---

(I (I (I $N_4$)))
≡ (I (I $N_5$))
≡ (I $N_6$)
≡ $N_7$

M is $x \to y \to ((x\ (A\ y))\ N_0)$ which really should be written as
$x \to y \to ((x\ ((z \to (z\ (n \to g \to y \to ((n\ g)\ (g\ y)))))\ y))\ (f \to x \to x))$

What is $((M\ N_3)\ N_4)$?

$$\equiv\ (((x \to y \to ((x\ (A\ y))\ N_0))\ N_3)\ N_4)$$

---

$$(((x \to y \to ((x\ (A\ y))\ N_0))\ N_3)\ N_4)$$
$$\equiv\ ((y \to ((N_3\ (A\ y))\ N_0))\ N_4)$$

---

$$((y \to ((N_3\ (A\ y))\ N_0))\ N_4)$$
$$\equiv\ ((N_3\ (A\ N_4))\ N_0)$$

---

$$((N_3\ (A\ N_4))\ N_0)$$
$$\equiv\ (((f \to x \to (f\ (f\ (f\ x))))\ (A\ N_4))\ N_0)$$

---

$$(((f \to x \to (f\ (f\ (f\ x))))\ (A\ N_4))\ N_0)$$
$$\equiv\ ((x \to ((A\ N_4)\ ((A\ N_4)\ ((A\ N_4)\ x))))\ N_0)$$

---

$$((x \to ((A\ N_4)\ ((A\ N_4)\ ((A\ N_4)\ x))))\ N_0)$$
$$\equiv\ ((A\ N_4)\ ((A\ N_4)\ ((A\ N_4)\ N_0)))$$

---

$$((A\ N_4)\ ((A\ N_4)\ ((A\ N_4)\ N_0)))$$
$$\equiv\ ((A\ N_4)\ ((A\ N_4)\ N_4))$$
$$\equiv\ ((A\ N_4)\ N_8)$$
$$\equiv\ N_{12}$$

$Sel_1$ is $x \to y \to x$
$Sel_2$ is $x \to y \to y$

so, $((Sel_1\ \mathcal{A})\ \mathcal{B}) \equiv \mathcal{A}$
and $((Sel_2\ \mathcal{A})\ \mathcal{B}) \equiv \mathcal{B}$, no matter what $\mathcal{A}$ and $\mathcal{B}$ are.

Pr is $x \to y \to s \to ((s\ x)\ y)$

so, $(Pr\ \mathcal{A}\ \mathcal{B})\ Sel_1 \equiv \mathcal{A}$
and $(Pr\ \mathcal{A}\ \mathcal{B})\ Sel_2 \equiv \mathcal{B}$, no matter what $\mathcal{A}$ and $\mathcal{B}$ are.

Swap is $x \to ((Pr\ (x\ Sel_2))\ (x\ Sel_1))$ or more formally
$x \to (((x \to y \to s \to ((s\ x)\ y))\ (x\ x \to y \to y))\ (x\ x \to y \to x))$

Step is x→((Pr (I (x $Sel_1$))) (x $Sel_1$))

or x→(((x→y→s→((s x) y)) (n→g→y→((n g) (g y)) (x x→y→x))) (x x→y→x))

(Step (Pr $N_0$ $N_0$))
≡ ((Pr (I ((Pr $N_0$ $N_0$) $Sel_1$))) ((Pr $N_0$ $N_0$) $Sel_1$))
≡ ((Pr (I $N_0$)) ((Pr $N_0$ $N_0$) $Sel_1$))
≡ ((Pr $N_1$) ((Pr $N_0$ $N_0$) $Sel_1$))
≡ ((Pr $N_1$) $N_0$)

(Step (Pr $N_1$ $N_0$))
≡ ((Pr $N_2$) $N_1$)

(Step (Pr $N_2$ $N_1$))
≡ ((Pr $N_3$) $N_2$)

(($N_5$ Step) $\mathscr{A}$)
≡ ((f→x→(f (f (f (f (f x))))) Step) $\mathscr{A}$)
≡ (x→(Step (Step (Step (Step (Step x))))) $\mathscr{A}$)
≡ (Step (Step (Step (Step (Step $\mathscr{A}$)))))

(($N_5$ Step) (Pr $N_0$ $N_0$))
≡ (Step (Step (Step (Step (Step (Pr $N_0$ $N_0$))))))
≡ (Pr $N_5$ $N_4$)

((($N_5$ Step) (Pr $N_0$ $N_0$)) $Sel_2$)
≡ ((Pr $N_5$ $N_4$) $Sel_2$)
≡ $N_4$

D is n→(((n Step) (Pr $N_0$ $N_0$)) $Sel_2$)

S is x→y→((y D) x)

(S $N_{12}$ $N_5$)
≡ (x→y→((y D) x) $N_{12}$ $N_5$)
≡ (($N_5$ D) $N_{12}$)
≡ ((f→x→(f (f (f (f (f x))))) D) $N_{12}$)
≡ (x→(D (D (D (D (D x))))) $N_{12}$)
≡ (D (D (D (D (D $N_{12}$)))))
≡ $N_7$

$$
\begin{array}{lllll}
T & \text{is} & Sel_1 & \text{which is} & x \to y \to x \\
F & \text{is} & Sel_2 & \text{which is} & x \to y \to y
\end{array}
$$

$$
\begin{array}{lll}
If & \text{is} & a \to a
\end{array}
$$

$$
\begin{array}{llll}
\text{so,} & (If\ T\ \mathcal{A}\ \mathcal{B}) & \equiv & (((If\ T)\ \mathcal{A})\ \mathcal{B}) \\
 & & \equiv & (((a \to a\ T)\ \mathcal{A})\ \mathcal{B}) \\
 & & \equiv & ((T\ \mathcal{A})\ \mathcal{B}) \\
 & & \equiv & ((x \to y \to x\ \mathcal{A})\ \mathcal{B}) \\
 & & \equiv & (y \to \mathcal{A}\ \mathcal{B}) \\
 & & \equiv & \mathcal{A}
\end{array}
$$

$$
\begin{array}{lll}
(If\ F\ \mathcal{A}\ \mathcal{B}) & \equiv & (((If\ F)\ \mathcal{A})\ \mathcal{B}) \\
 & \equiv & (((a \to a\ F)\ \mathcal{A})\ \mathcal{B}) \\
 & \equiv & ((F\ \mathcal{A})\ \mathcal{B}) \\
 & \equiv & ((x \to y \to y\ \mathcal{A})\ \mathcal{B}) \\
 & \equiv & (y \to y\ \mathcal{B}) \\
 & \equiv & \mathcal{B}
\end{array}
$$

$$
\begin{array}{lll}
Z & \text{is} & n \to ((n\ (y \to F))\ T)
\end{array}
$$

$$
\begin{array}{llll}
\text{so,} & (Z\ N_0) & \equiv & (n \to ((n\ (y \to F))\ T)\ N_0) \\
 & & \equiv & ((N_0\ (y \to F))\ T) \\
 & & \equiv & (((f \to x \to x)\ (y \to F))\ T) \\
 & & \equiv & ((x \to x)\ T) \\
 & & \equiv & T
\end{array}
$$

$$
\begin{array}{lll}
(Z\ N_1) & \equiv & (n \to ((n\ (y \to F))\ T)\ N_1) \\
 & \equiv & ((N_1\ (y \to F))\ T) \\
 & \equiv & (((f \to x \to (f\ x))\ (y \to F))\ T) \\
 & \equiv & ((x \to ((y \to F)\ x))\ T) \\
 & \equiv & ((x \to F)\ T) \\
 & \equiv & F
\end{array}
$$

$$
\begin{aligned}
(Z\ N_4) &\equiv (((f \to x \to (f\ (f\ (f\ (f\ x)))))\ (y \to F))\ T) \\
&\equiv ((x \to ((y \to F)\ ((y \to F)\ ((y \to F)\ ((y \to F)\ x)))))\ T) \\
&\equiv ((x \to ((y \to F)\ ((y \to F)\ ((y \to F)\ F))))\ T) \\
&\equiv ((x \to ((y \to F)\ ((y \to F)\ F)))\ T) \\
&\equiv ((x \to ((y \to F)\ F))\ T) \\
&\equiv ((x \to F)\ T) \\
&\equiv F
\end{aligned}
$$

And is $x \to y \to ((x\ y)\ F)$

so,

$$
\begin{aligned}
(And\ F\ F) &\equiv ((x \to y \to ((x\ y)\ F)\ F)\ F) \\
&\equiv (y \to ((F\ y)\ F)\ F) \\
&\equiv ((F\ F)\ F) \\
&\equiv ((x \to y \to y\ F)\ F) \\
&\equiv (y \to y\ F) \\
&\equiv F
\end{aligned}
$$

$$
\begin{aligned}
(And\ F\ T) &\equiv ((x \to y \to ((x\ y)\ F)\ F)\ T) \\
&\equiv (y \to ((F\ y)\ F)\ T) \\
&\equiv ((F\ T)\ F) \\
&\equiv ((x \to y \to y\ T)\ F) \\
&\equiv (y \to y\ F) \\
&\equiv F
\end{aligned}
$$

$$
\begin{aligned}
(And\ T\ F) &\equiv ((x \to y \to ((x\ y)\ F)\ T)\ F) \\
&\equiv (y \to ((T\ y)\ F)\ F) \\
&\equiv ((T\ F)\ F) \\
&\equiv ((x \to y \to x\ F)\ F) \\
&\equiv (y \to F\ F) \\
&\equiv F
\end{aligned}
$$

$$
\begin{aligned}
(And\ T\ T) &\equiv ((x \to y \to ((x\ y)\ F)\ T)\ T) \\
&\equiv (y \to ((T\ y)\ F)\ T) \\
&\equiv ((T\ T)\ F) \\
&\equiv ((x \to y \to x\ T)\ F) \\
&\equiv (y \to T\ F) \\
&\equiv T
\end{aligned}
$$

Not is x→((x F) T)

Or is x→y→((x T) y)

(Equal x y) ≡ (x-y is zero) and (y-x is zero)

Equal ≡ x→y→((And (Z ((S x) y))) (Z ((S y) x)))

(Less x y) ≡ (x-y is zero) and (y-x is not zero)

Less ≡ x→y→((And (Z ((S x) y))) (Not (Z ((S y) x))))

## The Paradoxical Combinator

Y is f→((x→(f (x x))) (x→(f (x x))))

(Y 𝒜) ≡ (f→((x→(f (x x))) (x→(f (x x)))) 𝒜)
≡ ((x→(𝒜 (x x))) (x→(𝒜 (x x))))

≡ ((x→(𝒜 (x x))) (x→(𝒜 (x x))))
≡ (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))
≡ (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))

≡ (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))
≡ (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x))))))
≡ (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x))))))

≡ (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x))))))
≡ (𝒜 (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))))
≡ (𝒜 (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))))

≡ (𝒜 (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x)))))))
≡ (𝒜 (𝒜 (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x))))))))
≡ (𝒜 (𝒜 (𝒜 (𝒜 ((x→(𝒜 (x x))) (x→(𝒜 (x x))))))))

(Y 𝒜) ≡ (𝒜 (Y 𝒜))

Note that Y is not recursive. (No lambda-expression can be)

This is not the factorial function, and it isn't recursive either:

$$Impf \equiv f\to(n\to(((If\ (Z\ n))\ N_1)\ ((M\ n)(f\ (D\ n)))))$$

To clarify it, I'll remove some brackets, replace (Z x) by x==0, replace ((M x) y) by x*y, and replace (D x) by x-1:

```
Impf(f)  =    n→ if (n==0)
                     return 1;
                 else
                     return n*f(n-1);
```

If f is an approximation to the factorial function, then Impf(f) will be a better approximation to the factorial function. For however many values of n that f(n) manages to calculate the correct factorial, Impf(f(n)) will get it right for at least one more value.

For example,

```
poor(x)  =    if (x==3) return 6;
              else if (x==4) return 24;
              else if (x==6) return 39;
              else if (x==7) return 5040;
              else return 73;
```

poor(x) only gets factorial right when x is 3, 4, or 7.
Impf(poor)(x) gets factorial right when x is 0, 4, 5, or 8.
Impf(Impf(poor))(x) gets factorial right when x is 0, 1, 5, 6, or 9.

```
stupid(x) = return -5;
```

stupid never gets factorial right, but
Impf(stupid) is correct for parameter 1,
Impf(Impf(stupid)) is correct for 0 and 1,
Impf(Impf(Impf(stupid))) is correct for 0, 1, and 2.
Impf(Impf(Impf(Impf(stupid)))) is correct for 0, 1, 2, and 3.

Remembering that

$$Y \text{ is } f\to((x\to(f\ (x\ x)))\ (x\to(f\ (x\ x))))$$

$$(Y\ \mathcal{A}) \equiv (\mathcal{A}\ (Y\ \mathcal{A}))$$

$$Impf \equiv f\to(n\to(((If\ (Z\ n))\ N_1)\ ((M\ n)\ (f\ (D\ n)))))$$

$$
\begin{aligned}
& ((Y\ Impf)\ N_5) \\
\equiv\ & ((Impf\ (Y\ Impf))\ N_5) \\
\equiv\ & (n\rightarrow(((If\ (Z\ n))\ N_1)\ ((M\ n)((Y\ Impf)\ (D\ n))))\ N_5) \\
\equiv\ & (((If\ (Z\ N_5))\ N_1)\ ((M\ N_5)((Y\ Impf)\ (D\ N_5)))) \\
\equiv\ & (((If\ F)\ N_1)\ ((M\ N_5)((Y\ Impf)\ (D\ N_5)))) \\
\equiv\ & ((M\ N_5)\ ((Y\ Impf)\ (D\ N_5))) \\
\equiv\ & ((M\ N_5)\ ((Y\ Impf)\ N_4))
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((M\ N_5)\ ((Impf\ (Y\ Impf))\ N_4)) \\
\equiv\ & ((M\ N_5)\ (n\rightarrow(((If\ (Z\ n))\ N_1)\ ((M\ n)((Y\ Impf)\ (D\ n))))\ N_4)) \\
\equiv\ & ((M\ N_5)\ (((If\ (Z\ N_4))\ N_1)\ ((M\ N_4)((Y\ Impf)\ (D\ N_4))))) \\
\equiv\ & ((M\ N_5)\ (((If\ F)\ N_1)\ ((M\ N_4)((Y\ Impf)\ (D\ N_4))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((Y\ Impf)\ (D\ N_4)))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((Y\ Impf)\ N_3)))
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((Impf\ (Y\ Impf))\ N_3)) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ (n\rightarrow(((If\ (Z\ n))\ N_1)\ ((M\ n)((Y\ Impf)\ (D\ n))))\ N_3)) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ (((If\ (Z\ N_3))\ N_1)\ ((M\ N_3)((Y\ Impf)\ (D\ N_3))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ (((If\ F)\ N_1)\ ((M\ N_3)((Y\ Impf)\ (D\ N_3))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((Y\ Impf)\ (D\ N_3)))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((Y\ Impf)\ N_2)))
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((Impf\ (Y\ Impf))\ N_2))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ (n\rightarrow(((If\ (Z\ n))\ N_1)\ ((M\ n)((Y\ Impf)\ (D\ n))))\ N_2))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ (((If\ (Z\ N_2))\ N_1)\ ((M\ N_2)((Y\ Impf)\ (D\ N_2)))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ (((If\ F)\ N_1)\ ((M\ N_2)((Y\ Impf)\ (D\ N_2)))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((Y\ Impf)\ (D\ N_2))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((Y\ Impf)\ N_1))))
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((Impf\ (Y\ Impf))\ N_1))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ (n\rightarrow(((If\ (Z\ n))\ N_1)\ ((M\ n)((Y\ Impf)\ (D\ n))))\ N_1))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ (((If\ (Z\ N_1))\ N_1)\ ((M\ N_1)((Y\ Impf)\ (D\ N_1)))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ (((If\ F)\ N_1)\ ((M\ N_1)((Y\ Impf)\ (D\ N_1)))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((M\ N_1)\ ((Y\ Impf)\ (D\ N_1))))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((M\ N_1)\ ((Y\ Impf)\ N_0))))
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((M\ N_1)\ ((Impf\ (Y\ Impf))\ N_0)))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ ((M\ N_1)\ (n\rightarrow(((If\ (Z\ n))\ N_1)\ \ldots)\ N_0))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ (((If\ (Z\ N_0))\ N_1)\ \ldots))) \\
\equiv\ & ((M\ N_5)\ ((M\ N_4)\ ((M\ N_3)\ ((M\ N_2)\ N_1)))) \\
\equiv\ & N_{120}
\end{aligned}
$$

Div(x, y) = if (x<y) then 0 else 1+Div(x-y, y)

Div takes two parameters, so it is a little more complicated than factorial. Define instead a function that takes a single parameter, which is a pair of numbers that happen to be x and y:

Div(p) = if (first(p)<second(p))
    then 0
    else 1+Div(pair(first(p)-second(p), second(p)))

Impd(d) = p → if (first(p)<second(p))
    then 0
    else 1+d(pair(first(p)-second(p), second(p)))

Impd ≡ d→p→(((If ((Less (p Sel$_1$) (p Sel$_2$))) N$_0$) (I (d ((Pr ((S (p Sel$_1$)) (p Sel$_2$))) (p Sel$_2$)))))

So,

| | |
|---|---|
| (Y Impd) | is a division function for pairs. |
| ((Y Impd) ((Pr x) y)) | calculates x divided by y. |
| x→(y→((Y Impd) ((Pr x) y))) | is integer division |

In its correct formal form, Divide is

x → ( y → ( ( ( f → ( ( x → ( f ( x x ) ) ) ( x → ( f ( x x ) ) ) ) ) ( d → p → ( ( ( ( a → a ) ( (
( x → ( y → ( ( ( x → ( y → ( ( x y ) ( x → ( y → y ) ) ) ) ) ) ( ( n → ( ( n ( y → ( a → ( b →
b ) ) ) ( a → ( b → a ) ) ) ) ) ( ( ( m → ( n → ( ( n ( n → ( ( ( n ( x → ( ( ( x → ( y → ( s →
( ( s x ) y ) ) ) ) ) ( n → ( g → ( y → ( ( n g ) ( g y ) ) ) ) ) ( x ( x → ( y → x ) ) ) ) ) ) ( x ( x → (
y → x ) ) ) ) ) ) ) ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) ( f → ( x → x ) ) ( f → ( x → x ) ) ) ) ) (
a → ( b → b ) ) ) ) ) ) m ) ) ) x ) y ) ) ) ( ( x → ( ( x ( a → ( b → b ) ) ) ( a → ( b → a ) ) ) ) )
( ( n → ( ( n ( y → ( a → ( b → b ) ) ) ( a → ( b → a ) ) ) ) ) ( ( ( m → ( n → ( ( n ( n → ( (
( n ( x → ( ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) ( n → ( g → ( y → ( ( n g ) ( g y ) ) ) ) ) ( x (
x → ( y → x ) ) ) ) ) ) ( x ( x → ( y → x ) ) ) ) ) ) ) ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) ( f → (
x → x ) ) ( f → ( x → x ) ) ) ) ) ( a → ( b → b ) ) ) ) ) ) m ) ) ) y ) x ) ) ) ) ) ) ) ( p ( x → ( y → x
) ) ) ( p ( x → ( y → y ) ) ) ) ) ) ( f → ( x → x ) ) ) ( ( n → ( g → ( y → ( ( n g ) ( g y ) ) ) ) ) (
d ( ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) ( ( ( m → ( n → ( ( n ( n → ( ( ( n ( x → ( ( ( x →
( y → ( s → ( ( s x ) y ) ) ) ) ) ( n → ( g → ( y → ( ( n g ) ( g y ) ) ) ) ) ( x ( x → ( y → x ) ) ) )
) ( x ( x → ( y → x ) ) ) ) ) ) ) ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) ( f → ( x → x ) ) ( f → ( x
→ x ) ) ) ) ) ( a → ( b → b ) ) ) ) ) ) m ) ) ) ( p ( x → ( y → x ) ) ) ) ) ( p ( x → ( y → y ) ) ) ) ) ) (
p ( x → ( y → y ) ) ) ) ) ) ) ) ) ) ) ( ( ( x → ( y → ( s → ( ( s x ) y ) ) ) ) ) x ) y ) ) )

The traditional notation is to use λa instead of a→
Church would have written

Y is λf(λxf(x(x))(λxf(x(x))))

rather than

Y is f→((x→(f (x x))) (x→(f (x x))))

The identifier that comes before an → is called a Bound Variable. An identifier that appears when it is not bound is called a Free variable.

In (a (b→((a b) (a→(a (b c))))))

b only appears as a bound variable
c only appears as a free variable
a is both bound and free.

A Lambda Expression is in Normal Form if reading from left to right, the bound variables all appear in alphabetical order, starting with a, never skipping any, and never unnecessarily using a new bound variable.

(d ((a→b→a) (a→b→(x c→a)))) is normal
((a→a) (b→b)) is not normal, its normal form is ((a→a) (a→a))

A Lambda Expression is in Ground Form if no β-reductions are possible. That is, it contains nothing like ((a→B) C) anywhere within it.

Notice that in this

(Y $\mathcal{A}$) ≡ (f→((x→(f (x x))) (x→(f (x x)))) $\mathcal{A}$)

there are two possible β-reductions that could be performed first.

## Church-Rosser Theorems

1. Starting from any given lambda expression, every possible ordering of α- and β-reductions that leads to a Ground Normal Form will lead to the same Ground Normal Form.

2. Starting from any given lambda expression, if it is possible to reach a Ground Normal Form, then a strategy of always choosing the left-most possible β-reduction first is guaranteed to reach a Ground Normal Form. This strategy is called Normal Order.