Harmonic Analysis of Nonlinear Devices for Virtual Bass System

Nay Oo, Woon-Seng Gan
Digital Signal Processing Lab, School of EEE, Nanyang Technological University, Singapore
{e070001, ewsgan}@ntu.edu.sg

Abstract

Nonlinear devices (NLDs) are commonly used to generate specific harmonics in the virtual bass system. However, a detailed mathematical analysis of these NLDs in virtual bass system is still lacking. In this paper, a single tone and two tones mathematical harmonic analysis of polynomial and exponential typed NLDs are presented. Mathematical tools such as Taylor Series and Chebyshev Polynomials are used to derive a mathematical series of harmonic generated from NLDs. MATLAB numerical calculations were performed using the harmonic analysis equations. Total Harmonic Richness (THR) values were calculated. Region of dominance from psychoacoustic pitch perception findings were related with this paper findings to show the construction of tailor-made NLDs using polynomial nonlinearities.

1. Introduction

Nowadays, audio enabled consumer electronic devices are getting smaller in size. The loudspeakers embedded in these portable devices are also becoming smaller to suit the form factor of the device, and without degrading the audio quality too much. However, small loudspeakers have physical limitation that they cannot radiate enough low frequency (bass) sound waves, and are often perceived as lack of bass.

In audio engineering, there has always been a strong interest in reproducing good bass frequencies [1-3]. The fundamental question is how to extend the low frequency bandwidth of the audio reproduction devices, such as small loudspeakers and low-cost earphones. To extend the low frequency audio bandwidth, we can make use of pitch perception capability of our human ears to create the virtual bass in the human auditory system by means of psychoacoustic signal processing [1-3].

Nonlinear device (NLD) is the key component in most of the virtual bass signal processing systems [1-3]. The objective of having an NLD in the virtual bass system is to generate the harmonics from fundamental frequencies of the input signals. These generated harmonics create virtual pitch at the place of fundamental frequency component. Even if the fundamental component is removed, human can still perceive the virtual pitch at the fundamental component based on psychoacoustic phenomenon, known as the “missing fundamental” [1-3][6-8].

Although harmonics are useful for virtual bass system, there are other undesired components such as intermodulation frequencies which are also produced by the NLD. In this paper, harmonic analysis equations used for both harmonic and intermodulation components are presented. This paper is organized as follows. Section 2 presents the analysis equations and simulation results of the single tone harmonic analysis. Section 3 presents the two tones harmonic analysis for intermodulation components. Section 4 summaries the key results in this paper and provide insights into how nonlinear functions are chosen and how NLDs can be constructed from polynomial functions based on the recent and previous findings from psychoacoustics.

2. Single tone harmonic analysis

In this section, the single tone harmonic analysis of the n^{th} order static nonlinear system (nonlinearity) is examined.

2.1. Generalized single tone harmonic analysis

The polynomial based NLD single tone analysis is based on the works by Schaefer [4]. He derived the mathematical relationship between polynomial power series (1) and Fourier series (2) using a relationship from Chebyshev polynomial of the first kind.

$$y = f(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots \quad (1)$$

$$y = \frac{1}{2} c_0 + c_1 \cos\theta + c_2 \cos 2\theta + c_3 \cos 3\theta + \dots, \quad (2)$$

where y and x denotes the output and input to the NLD. A single tone cosine wave is fed into the NLD for the harmonic analysis. Fourier coefficients can then be computed from polynomial series coefficients. Based on this idea, Schaefer derived a generalized equation which produces Fourier series (2) coefficients $\{c_0, c_1, c_2, c_3, ...\}$ from the polynomial series coefficients $\{h_0, h_1, h_2, h_3, ...\}$. The Schaefer's equation for a single tone harmonic analysis is shown as follows:

$$c_k = \frac{1}{2^{k-1}} \sum_{j=0}^{\infty} \frac{h_{k+2j}}{2^{2j}} \binom{k+2j}{j}, \tag{3}$$

where binomial coefficients are

$$\binom{k+2j}{j} = \frac{(k+2j)!}{(k+j)!(j!)}. \tag{4}$$

In this paper, we denote a term known as the total harmonic richness (THR) which is a ratio between the powers of NLD generated harmonics (from 1st to 6th order) to the power of the fundamental input tone. We included only the first six harmonics due to the findings from region of dominance in psychoacoustic pitch perception researches [6-8]. The THR formula for single tone analysis is

$$THR = \frac{(H_1)^2 + (H_2)^2 + \cdots + (H_6)^2}{(ST)^2}, \tag{5}$$

where the numerator is the summation of the power of the first six harmonics, and the denominator is the power of the fundamental input tone. $\{H_1, H_2, \ldots, H_6\}$ are the magnitudes of the generated harmonics from the NLD and ST is the magnitude of the input sine tone, set as unity as for this paper. The simulation results are in Table 1.

Table 1. Single tone harmonic analysis results (THR) of nonlinearities of different orders.

n	THR	%
2	0.250000	25.00
3	0.625000	62.50
4	0.265625	26.56
5	0.492188	49.22
6	0.255859	25.59
7	0.418701	41.87
8	0.243164	24.32
9	0.369690	36.97
10	0.230885	23.09

2.2. Exponential function, b^x

In this section, the single tone harmonic analysis of the exponential function is presented. The base constant b is set to e, the exponential function can be expanded to a polynomial function by using the Taylor's series.

$$y = f(x) = e^x = \sum_{k=0}^{\infty} \frac{(x)^k}{k!}. \tag{6}$$

Schaefer has previously derived the generalized single tone harmonic analysis equation for this exponential function based on his works on harmonic analysis for the bipolar junction transistor [5]. In this paper, we modify his equation for the analysis of exponential function used in the virtual bass system. The modified Schaefer's second equation for the single tone harmonic analysis of exponential function, $y = e^x$ is stated as

$$c_k = \left(\frac{1}{2}\right)^k \sum_{j=0}^{\infty} \frac{(1/2)^{2j}}{j!(k+j)!}, \tag{7}$$

where c_k is the magnitude of the k^{th} harmonic, generated by the exponential function.

However, the exponential function of $y = e^x$ has base constant, e which cannot be varied as a parameter to adjust the THR values if it is required. Therefore, we modified (6) and investigated the characteristic of varying the base constant of the exponential function and how base constant influences the THR.

Since $b^x = e^{x \ln b}$ and using (6), the exponential function with different base function can be converted to the polynomial form as follows,

$$b^x = \sum_{k=0}^{\infty} \frac{(x \ln b)^k}{k!}. \tag{8}$$

Therefore, the generalized single tone harmonic analysis formula for b^x can be derived from (7), and the k^{th} generated components from the exponential function with variable base, b can be derived as

$$c_k = \left(\frac{\ln b}{2}\right)^k \sum_{j=0}^{\infty} \frac{(\ln b/2)^{2j}}{j!(k+j)!}. \tag{9}$$

Using (9), a MATLAB program was written to calculate the THR values of different base, b. The results are listed in Table 2.

From Table 2 results, the following observations are made.

1. THR increases with the exponential base, b.
2. By increasing the base too much will result in strong harmonic components at the output and may cause saturation problem in real-time implementation due to the harmonic amplitudes which are larger than one.
3. Therefore, by adjusting the base, b, the desired THR value can be obtained.

The measurement of THR is the first assessment of audio nonlinearity for NLD. However, it alone cannot predict the entire performance of the nonlinear system. There are other generated undesired components called intermodulation frequencies which are heard as unpleasant audio distortion. In the following section, the intermodulation components, generated by NLD, are examined.

Table 2. Single tone harmonic analysis results (THR) of the exponential function, $y = b^x$.

b	THR	%
1.5	0.173020	17.30
2.5	1.083776	108.38
3.5	2.503114	250.31
4.5	4.376454	437.65
5.5	6.686533	668.65
6.5	9.423839	942.38
7.5	12.581751	1258.18
8.5	16.155135	1615.51
9.5	20.139775	2013.98

3. Two tones harmonic analysis

To examine the intermodulation components, the two tones of 100 Hz and 120 Hz are fed into the NLD. To quantify the performance of the different orders of nonlinearities, the following three metrics are used:

$$\Delta_H = \frac{\sum_{k=1}^{L} H_k^2}{\sum_{i=1}^{2} T_i^2}, \tag{10}$$

$$\Delta_{IM} = \frac{\sum_{k=1}^{M} IM_k^2}{\sum_{i=1}^{2} T_i^2}, \tag{11}$$

$$HIDR = \frac{\sum_{k=1}^{L} H_k^2}{\sum_{k=1}^{M} IM_k^2}, \tag{12}$$

where Δ_H denotes the ratio from the total power of interharmonic components to the total power of two input tones. Δ_{IM} denotes the ratio from the total power of intermodulation components to the total power of two input tones. $\sum_{k=1}^{M} IM_k^2$ is the summation of the power of M intermodulation components, generated by NLD or a particular nonlinearity under investigation. $\sum_{k=1}^{L} H_k^2$ is the summation of the power of L harmonic components generated by the NLD. $\sum_{i=1}^{2} T_i^2$ is the total power of the two input tones. HDR is the Harmonic Distortion energy Ratio measuring the ratio between the generated harmonics' powers to the generated intermodulation distortion components' powers.

To simulate the intermodulation effect, the "interharmonic analysis formula" for nonlinear system identification [9] was modified to suit the two tones harmonic analysis. $X(j\omega)$ denotes two input tones and $Y(j\omega)$ denotes the nonlinearity output components of DC, fundamental frequencies, harmonics of even and odd order and intermodulation components. They can be expressed as

$$X(j\omega) = \sum_{k=1}^{4} A(k) e^{j\phi(k)} \delta(\omega - \omega_0(k)) \tag{13}$$

with

$A(2+l) = A(l)$, $\omega_0(2+l) = -\omega_0(l)$, $\phi(2+l) = -\phi(l)$, $l = \{1,2\}$,

where A, ω_0 and ϕ are amplitude, frequency and phase. The output of the second order nonlinearity can be expressed as

$$Y(j\omega) = \sum_{n=1}^{4} \sum_{m=1}^{4} [A(n)A(m)] e^{j(\phi(n)+\phi(m))} \times \delta(\omega - [\omega_0(n) + \omega_0(m)]). \tag{14}$$

Extending the expansion equation, the frequency combination equation for an n^{th} order nonlinearity can be constructed with n nested summations. A simulation was performed up to 6^{th} order nonlinearity. The simulation framework is shown in Fig. 1, and the results are in Table 3.

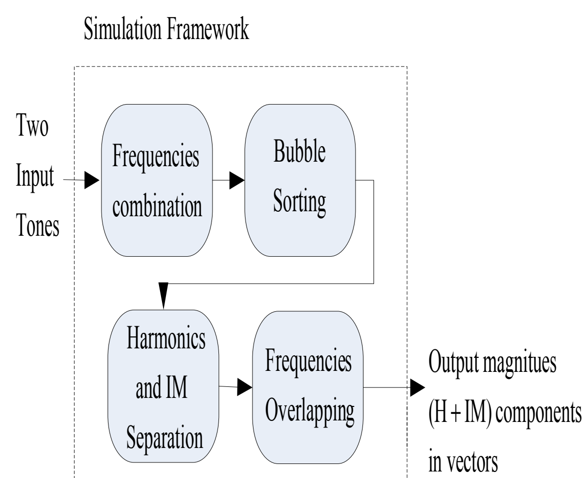

Fig. 1. Simulation framework for two tones harmonic analysis.

Table 3. Simulation results of Δ_H and Δ_{IM} for the different orders of nonlinearities.

n	Δ_H	Δ_{IM}	$HIDR$
2	0.2500	1.0000	0.250000
3	5.1250	1.1250	4.555556

4	4.0156	10.0625	0.399068
5	41.5078	20.5078	2.024000
6	50.7402	118.6387	0.427687

In Table 3, n denotes the order of nonlinearity. Two input tones of 100 Hz and 120 Hz are fed into the nonlinearity. From the computed values, Δ_H , Δ_{IM} and *HIDR* values are calculated with respect to the nonlinearity order. The following observations are noted.

1. Δ_H and Δ_{IM} go higher with the increasing order of nonlinearities.
2. $\Delta_H > \Delta_{IM}$ when the order of nonlinearity is odd, but $\Delta_H < \Delta_{IM}$ when the order of nonlinearity is even.
3. *HIDR* of odd order nonlinearity is always greater than the even order ones

These findings can be interpreted as all the even order nonlinearities generate DC component which takes up a large amount of output energy. DC components are not taken into the calculation of Δ_H , Δ_{IM} and *HIDR* since they are not useful for the harmonic generation.

3.1. Exponential function, b^x

Since the exponential function with base b can be approximated using the polynomial series (8), it can be viewed as a weighted sum of static nonlinear systems (nonlinearities). By using the simulation framework of Fig. 1, the Δ_H , Δ_{IM} and *HIDR* for the two input tones can be computed. The results are listed in Table 4. The highest nonlinearity order is set to six. The base constant is varied from 1.5 to 9.5 and the numerical results are obtained. Using (8), the exponential function b^x can be approximated as

$$b^x = \sum_{k=0}^{6} \{\frac{(\ln b)^k}{k!}\} x^k \tag{15}$$

where $h_k = \frac{(\ln b)^k}{k!}$ are the weights of the nonlinearities. Based on this approximation, the exponential function NLD, b^x can be constructed using polynomials NLD or blocks of static nonlinear systems.

It is observed that Δ_H and Δ_{IM} increase accordingly with increasing base, b of the exponential function, as shown in Table 4. In each row, Δ_H and Δ_{IM} are approximately the same, meaning that the interharmonic distortion contribution and intermodulation distortion contribution are balanced for the exponential function NLD. *HIDR* in table 4 shows that the higher the base is, the higher *HIDR* value is obtained, meaning that harmonics' energy with respect to the intermodulation components' energy is increased with increasing base values. However, varying the base too much has a danger of harmonics and intermodulation components over amplification which is generally not desirable.

Table 4. Simulation results of Δ_H and Δ_{IM} for b^x .

b	Δ_H	Δ_{IM}	*HIDR*
1.5	0.0023	0.0069	0.336856
2.5	0.1330	0.2041	0.651773
3.5	0.7754	0.8595	0.902182
4.5	2.3346	2.2139	1.054528
5.5	5.1759	4.5577	1.135636
6.5	9.6435	8.2206	1.173083
7.5	16.0676	13.5530	1.185538
8.5	24.7655	20.9135	1.184188
9.5	36.0396	30.6602	1.175451

4. Discussions

The objective of the virtual bass system is to extend the low frequency bandwidth of the small loudspeakers by making use of NLD, which can generate harmonics from input audio signal. These generated harmonics can create virtual pitch in the human auditory system. Even if the fundamental of the original signal, which falls below the loudspeaker cut-off frequency, is removed, the pitch at the location of the fundamental frequency is still perceived. By making use of this well-known psychoacoustics phenomenon, known as the "missing fundamental", and combining with the nonlinear system theory, we can design the NLD for virtual bass enhancement system.

Virtual bass system shifts the low frequencies, which are below the small loudspeaker's cut-off frequency, to the mid-range by NLD-generated harmonics. These harmonics in the mid-range create perceived virtual bass in the auditory system. The question on which harmonics are the most important for pitch perception is known as determining dominance region in the psychoacoustic literature [6-8]. However, there is no exact agreement between researchers on which harmonics are the most important based on subjective listening tests. However, all reported generally that harmonics from one to six are important when the fundamental frequency is below 1400 Hz.

To construct a NLD using individual nonlinearities with weighted gains, the sixth order nonlinearity is enough to generate the harmonics number up to six which are in the dominance region of perceived pitch.

Hence, the infinite series (8) can be approximated up to the sixth order in (15) without loss of precision and psychoacoustic requirement.

In this paper, another kind of NLD called the exponential NLD is formulated and simulated. Since the exponential function can be approximated using Taylor polynomial series, it can be said that the exponential function NLD is a special kind of polynomial NLD. As for the polynomial NLD, the design parameters are the individual gains of the nonlinearity blocks whereas for the exponential NLD, the design parameter is the base, b .

Table 2 shows that increasing the base of the exponential function results in dramatic increase in THR values. In addition, Table 4 shows that increasing the base also results in increasing the energy of interharmonic and intermodulation components if they are present. However, Table 4 shows that the energy ratio between the generated harmonics, which are desirable, and the intermodulation components, which are undesirable, are increased with increasing base. This can be interpreted as increasing the base of the exponential function will result in stronger harmonics and intermodulation components but the harmonics energy are stronger than intermodulation components energy.

The infinite series of harmonics can be generated by the exponential function NLD. The generated harmonic level decays as a function of the amplitude of the fundamental component, which is a desired response for natural musical instruments [3]. If the input level is high, the exponential NLD can generate stronger harmonics. Otherwise, it generates weaker harmonics. That is why the exponential NLD is a good NLD to be used in virtual bass system.

By using Taylor's series, the exponential NLD can be approximated to the polynomial form. If we want to implement the exponential function NLD as in left hand side of equation (15), we have the base, b as the parameter to adjust the harmonic richness. On the other hand, the same exponential function NLD in the right hand side of equation (15), we have the weights, h_k to parameterize the NLD. In addition, by knowing the facts of static nonlinearities simulation results, described earlier, we can also connect the nonlinearities blocks up to sixth order with adjustable weights to design the tailor-made NLD for the virtual bass enhancement system.

5. Conclusions

Harmonic analysis of single tone and two tones for the NLDs used in virtual bass system was presented in this paper. Mathematical harmonic analysis formulas were used to develop the simulation framework. This simulation framework was then used to obtain the THR , Δ_H , Δ_{IM} and $HIDR$ values for single tone and two tones NLD harmonic analysis. Results were obtained, summarized and related with psychoacoustic pitch perception researches to study the effect of polynomial NLD and exponential NLD which can be used in the virtual bass system.

6. Acknowledgement

The authors would like to acknowledge the support of National Research Foundation (NRF) for Interactive Digital Media, under the Media Development Authority for funding this work.

7. References

[1] E. Larsen and R. M. Aarts, “Reproducing Low-Pitched Signals through Small Loudspeakers”, *J Audio Eng. Soc.*, Vol. 50, No. 3, (March 2002), pp. 147-164.

[2] W. S. Gan, S. M. Kuo and C. W. Toh, “Virtual Bass for Home Entertainment, Multimedia PC, Game Station and Portable Audio Systems”, *IEEE Trans. Consumer Electronics*, Vol 47, (2001), pp. 787-794.

[3] F. A. Karnapi, “Psychoacoustic Signal Processing Techniques in Bass Perception Enhancement”, *M.Eng. Thesis,* School of EEE, Nanyang Technological University, (2002).

[4] R. A. Schaefer, “Electronic Musical Tone Production by Nonlinear Waveshaping”, *J. Audio Eng. Soc.*, Vol.18, No. 4, (1970), pp.413-417.

[5] R. A. Schaefer, “Production of Harmonics and Distortion in p-n Junctions”, *J Audio Eng. Soc.*, Vol 19, No. 9, (1971), pp. 759-768.

[6] H. Dai, "On the relative influence of individual harmonics on pitch judgement", J. Acoust. Soc. Am., Vol. 107, No. 2, (February 2000), pp. 953-959.

[7] Brian C. J. Moore, B. R. Glasberg and R. W. Peters, “Relative dominance of individual partials in determining the pitch of complex tones”, *J. Acoust. Soc. Am.*, Vol . 77, No. 4, (May 1995), pp. 1853-1860.

[8] R. J. Ritsma, "Frequencies dominant in the perception of the pitch of complex sounds*", J. Acoust. Soc. Am.*, Vol. 42, No. 1, (1967), pp. 191-198.

[9] C. Evans, D. Rees and L. Jones, “Nonlinear Disturbance Errors in System Identification Using Multisine Test Signals”, IEEE Trans. On Instru. And Measurement, Vol. 43, No. 2, (1994), pp 238-24.