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Harmonic and Intermodulation Analysis of Nonlinear Devices Used in Virtual Bass Systems

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ABSTRACT

Nonlinear devices (NLD) are used in the virtual bass system. NLD generates harmonics which in turn create the pitch perception and are used in the audio bass enhancement systems using psychoacoustics. This paper presents the mathematical derivations and analyses of five different NLD devices, together with intermodulation analysis of harmonics generated by these NLDs. The five NLDs are half-wave rectifier, full-wave rectifier, square wave, polynomial function and exponential function. The derivation of harmonic analysis equations are based on Fourier Theorems, Chebyshev Polynomials, and Taylor Series expansions. Besides the harmonics, intermodulation components are also resulted from NLDs. Both mathematical analysis and simulation results are presented for the intermodulation effects of harmonics generated by NLDs.

1.INTRODUCTION

Small loudspeakers, embedded in the consumer portable electronic devices, cannot reproduce rich bass frequencies, because of psychical size limitation. Similarly, low-quality headphones also have very poor bass frequency response. Therefore, the audio reproduction bandwidths of these devices are very limited. Normally, bass frequencies are below 250 Hz, and the small loudspeakers' or low-quality headphones' cut-off frequencies are higher than 250 Hz. The audio frequencies below the cut-off frequencies cannot be reproduced or attenuated severely.

To extend the low-frequency audio bandwidth, without pushing the physical limit of audio reproduction devices, we can make use of a psychoacoustic phenomenon, called the "Missing Fundamental" [10-12]. The "Missing Fundamental" phenomenon states that human can perceive the virtual pitch at the fundamental frequency when the harmonics are present, even if the fundamental frequency itself is removed or not present. This phenomenon can be used to create the virtual bass system or low frequency psychoacoustic audio bandwidth extension systems [1-9].

This topic has been well-researched by various researchers, both in academic and industry. A variety of bass enhancement systems using psychoacoustics have been implemented, and reported in the literatures [1-9]. Basically, there are two approaches to implement the psychoacoustic bandwidth extension system or virtual bass system, such as time-domain approach and frequency-domain approach [21]. In time-domain approach, NLD is used as a central processing block to generate the harmonics based on incoming audio signal. By generating harmonics, the virtual bass system can create virtual pitch at the bass frequencies, which our human ear can perceive, even though those frequencies may not be psychically present [1-9].

Since NLD is used in the central processing block of the time-domain approach virtual bass system, five types of NLDs have been studied in this paper. Harmonic analysis using single tone as an input and intermodulation distortion analysis using logarithmic multitones as inputs are also investigated. Virtual bass system research is an interdisciplinary research, containing psychoacoustics, non-linear system theory and signal processing. In this paper, we attempted to relate these three fields to understand the nature of NLDs for virtual bass system. Therefore, in Section 2, we present the harmonic and intermodulation distortion analysis of static memoryless nonlinearities up to sixth order polynomial. These polynomials can in turn be viewed as a parallel connection of static memoryless nonlinearities as shown in Figure 1 [22-23]. Section 3 presents the detailed analysis of five types of NLDs, including harmonic analysis and intermodulation distortion analysis. Section 4 discusses the NLD simulation results. Section 5 concludes this paper.

2. STATIC MEMORYLESS NONLINEARITIES

NLD is a nonlinear device which can be implemented either digital or analog means. It can be constructed using dynamical systems which have memory or static memoryless nonlinearities [22]. The five NLDs presented in this paper are static memoryless nonlinear systems and can be constructed as a parallel connection of different orders of weighted nonlinearities, as shown in Figure 1.

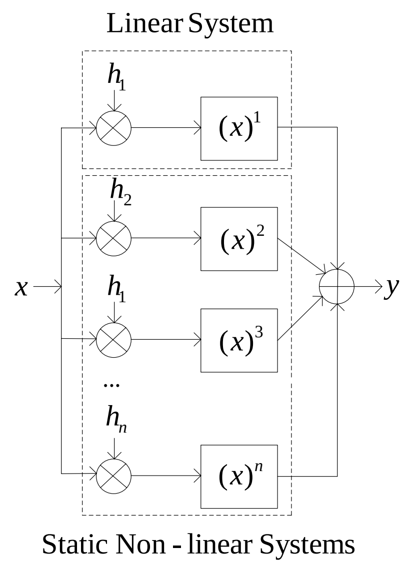

Figure 1: A block diagram showing a polynomial NLD can be constructed using different orders of nonlinearities, connected in parallel.

The usefulness of nonlinear system for the virtual bass system is that it can generate new frequencies, which is not possible for linear system. The nonlinear system generates new frequencies as harmonics, that consist of desired components to enhance the bass in virtual bass system, and intermodulation components that are undesired components and cause unpleasant distortion. To understand the nature of harmonic generation and intermodulation contamination, we can use two tests. The first test is the single tone test and the second test is the multitones test. Subsequent sections present the single tone harmonic analysis, multitones harmonic and intermodulation component analysis, and four measurement metrics to perform objective comparisons among the NLDs. All the results presented in this paper are obtained using MATLAB.

2.1. Single Tone Harmonic Analysis

A single sine tone with an adjustable amplitude is fed into the NLD (or nonlinearity) under investigation, as shown in Figure 2. The output harmonics amplitudes are measured using Discrete Fourier Transform (DFT) or respective derived mathematical formulas, such as Schaefer-Suen equations.

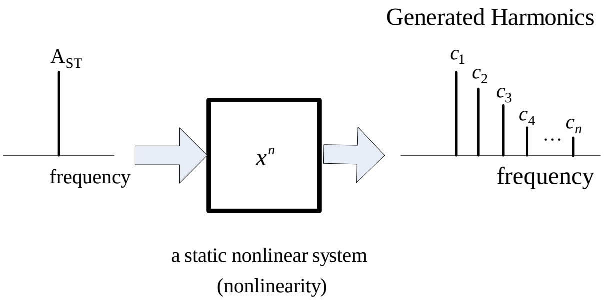

Figure 2: A diagram showing the single tone input harmonic analysis of a static nonlinear system.

As described in previous section, any static memoryless nonlinear system can be approximated by polynomial series. Referring to Figure 1, nonlinearity x^n is a sub block of polynomial NLD. The polynomial based NLD single tone analysis is based on the works by R. A. Schaefer [14] and C. Y. Suen [24]. Schaefer derived the mathematical relationship between polynomial power series (1) and Fourier series (2) using a relationship from Chebyshev polynomial of the first kind [15].

$$y = f(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots \tag{1}$$

$$y = \frac{1}{2} c_0 + c_1 \cos\theta + c_2 \cos 2\theta + c_3 \cos 3\theta + \dots , \tag{2}$$

where y and x denotes the output and input of the NLD, respectively. A single tone cosine wave is fed into the NLD for the harmonic analysis. This can be expressed mathematically as follows.

$$x = \cos\theta \, , \theta = 2\pi f t \, , \tag{3}$$

where f is the frequency in Hz, and t is the time in seconds. If $\cos\theta$ is fed into the NLD in (1), output becomes

$$\begin{aligned} y &= f(x) \\ &= f(\cos\theta) \\ &= h_0 + h_1 \cos\theta + h_2 \cos^2\theta + h_3 \cos^3\theta + \dots \end{aligned} \tag{4}$$

Equation (2) and (4) can be related by Chebyshev polynomials of the first kind:

$$T_k(\cos\theta) = \cos k\theta \, . \tag{5}$$

Fourier coefficients can then be computed from polynomial series coefficients. Base on this idea, Schaefer derived a generalized equation which produces Fourier series (2) coefficients $\{c_0, c_1, c_2, c_3, ...\}$ from the polynomial series coefficients $\{h_0, h_1, h_2, h_3, ...\}$. This equation is powerful in the sense that it can calculate the magnitudes of harmonics produced by any order of static nonlinearities individually, as well as any order of polynomial based NLD, excited by a single tone. The Schaefer's equation for a single tone harmonic analysis is shown as follows:

$$c_k = \frac{1}{2^{k-1}} \sum_{j=0}^{\infty} \frac{h_{k+2j}}{2^{2j}} \binom{k+2j}{j} , \tag{6}$$

where binomial coefficients are

$$\binom{k+2j}{j} = \frac{(k+2j)!}{(k+j)!(j!)} \, . \tag{7}$$

However, Schaefer's original equation has one limitation that it fixes the input tone to unity. To overcome this limitation, C. Y. Suen [24] derived a generalized harmonic analysis equation with varying single tone amplitude, A as a parameter using a different approach. Suen's equation took into consideration of the adjustable input amplitude tone. By comparing the two equations and rearranging the terms, we arrive at the Schaefer-Suen equation for the generalized harmonic equation as

$$c_k = \frac{1}{2^{k-1}} \sum_{j=0}^{\infty} A^{k+2j} \times \frac{h_{k+2j}}{2^{2j}} \binom{k+2j}{j} . \tag{8}$$

With (8), we can adjust the amplitude of the input tone, A and calculate the generated harmonic amplitudes directly from polynomial coefficients, h_k's. The infinite series in (8) is convergent, and for the nth order nonlinearity, we only need to calculate up to $k = n$ or $c_{k=n}$. This equation is powerful in the sense that without going through DFT, we can calculate the generated harmonics amplitudes using simple algebraic formulas. Using (8), we can also count the total number of harmonics generated by a particular nonlinearity and list the results in Table 1 as follows.

Table 1. Relationship between harmonic numbers, total number of harmonics and order of nonlinearity.

n	DC and Harmonics Numbers	Total number of harmonics, H
2	DC, 2	1
3	1,3	2
4	DC, 2, 4	2
5	1,3,5	3
6	DC, 2, 4, 6	3
7	1,3,5,7	4
8	DC, 2, 4, 6, 8	4
9	1,3,5,7,9	5
10	DC, 2, 4, 6, 8, 10	5

From Table (1), the following observations are made.

- The number of harmonics produced by the nonlinearity increases when the order of nonlinearity increases.
- The odd order nonlinearity can produce only odd harmonics, and the even order nonlinearity can produce only even order harmonics.
- The odd order nonlinearity always reproduces the fundamental which is the first harmonic number.
- The even order nonlinearity always produces DC component.
- The maximum harmonic number is always equal to the order of nonlinearity.
- For the total number of harmonics generated, H can be formulated as follows:

$$H = \frac{n}{2} \quad (n \text{ is even}), \tag{9}$$

$$H = \frac{n+1}{2} \quad (n \text{ is odd}). \tag{10}$$

2.2. Multitones Harmonic Analysis

In a virtual bass system, the generated harmonics are desired frequency components which create virtual bass perception in the human auditory system, whereas generated intermodulation components are heard as audio distortion. The intermodulation components are formed by addition or subtraction of two or more input frequency components, as shown in Figure 3.

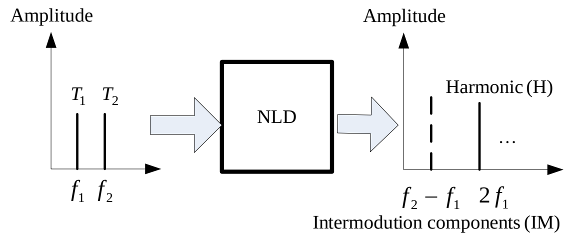

Figure 3: A diagram showing the two input tones intermodulation analysis.

In virtual bass system, the input signal to the NLD are low-pass filtered [1-6]. The cut-off frequency of the low-pass filter is near the loudspeaker cut-off frequency or resonance frequency. Therefore, the input signal spectrum to the NLD is assumed to be less than 200 Hz (audio bass frequencies). From 20 Hz to 200 Hz, we calculate five logarithmic tones as follows [22]. We use logarithmic-multitones stimulus as input because previous research findings by others showed that only logarithmically equal space multitones can generate more intermodulation distortion components of the device under test [22]. The calculation of the frequencies of five logarithmic multitones is listed in Table 2.

Table 2. Five logarithmic multitones calculations from 20 Hz to 200 Hz.

No.	Calculation	Rounded
1	20 Hz	20 Hz
2	$20 \times \log^{-1}(1/4) = 35.565$ Hz	36 Hz
3	$35.565 \times \log^{-1}(1/4) = 63.246$ Hz	63 Hz
4	$63.246 \times \log^{-1}(1/4) = 112.468$ Hz	112 Hz
5	$112.468 \times \log^{-1}(1/4) = 200$ Hz	200 Hz

To simulate the intermodulation effect, the “interharmonic analysis formula” for nonlinear system identification was modified to suit the two tones harmonic analysis [20]. $X(j\omega)$ denotes two input tones, and $Y(j\omega)$ denotes the nonlinearity output components of DC, fundamental frequencies, harmonics of even and odd order and intermodulation components. They can be expressed as

$$X(j\omega) = \sum_{k=1}^{10} A(k) e^{j\phi(k)} \delta(\omega - \omega_0(k)) \tag{11}$$

with

$$A(5+l) = A(l)\,,$$
$$\omega_0(5+l) = -\omega_0(l)\,, \tag{12}$$
$$\phi(5+l) = -\phi(l)\,,$$
$$l = \{1,2,3,4,5\}\,,$$

where A, ω_0 and ϕ are amplitude, frequency and phase. The output of the second order nonlinearity can be expressed as

$$Y(j\omega) = \sum_{n=1}^{10}\sum_{m=1}^{10} [A(n)A(m)] e^{j(\phi(n)+\phi(m))} \times \tag{13}$$
$$\delta(\omega - [\omega_0(n) + \omega_0(m)]).$$

The output of the third order nonlinearity can also be expressed as

$$Y(j\omega) = \sum_{n=1}^{10}\sum_{m=1}^{10}\sum_{k=1}^{10} [A(n)A(m)A(k)] e^{j(\phi(n)+\phi(m)+\phi(k))} \tag{14}$$
$$\delta(\omega - [\omega_0(n) + \omega_0(m) + \omega_0(k)])\,.$$

Extending the expansion equation, the frequency combination equation for an n^{th} order nonlinearity can be constructed with n nested summations. A simulation was performed up to 6^{th} order nonlinearity. The simulation framework is shown in Figure 4 which can be used for simulating any nonlinearity block in Figure 1. The results of individual nonlinear system simulation can be combined together to construct the NLDs simulation. As for this later case, individual nonlinearities are simulated first and the resultant harmonics are combined to get the overall system response. For the first stage, *Frequency Combination*, the total number of frequency contributions, C [20] can be formulated as

$$C = (2F)^n\,, \tag{15}$$

where F is the number of multitones inputs and n is the order of nonlinearity. In this paper, five logarithmic tones $(F = 5)$, as shown in Table 2, are used. The frequency combinations for the first stage can be calculated, as described in Table 3. The second stage, *Sorting*, sorts the generated frequencies in ascending order. At this stage, frequencies are not overlapping yet. This stage can take up a long time if not an efficient algorithm is used. A MATLAB command, *sortrows*, is used in our simulation. The third stage, *Harmonic and Intermodulation Components Separation* decouples harmonics and intermodulation frequencies and put into the separate vectors. By this stage, the harmonic components and intermodulation distortion components can be found. The final stage, *Frequency Overlapping* overlaps the redundant frequencies. At the end of the process, we can obtain two vectors with harmonics and intermodulation components separately.

Table 3. Frequency combinations and order of nonlinearity for the first stage of Figure 4.

Nonlinearity Order, n	Total Number of Frequency Combinations, C
2	100
3	1000
4	10000
5	100000
6	1000000

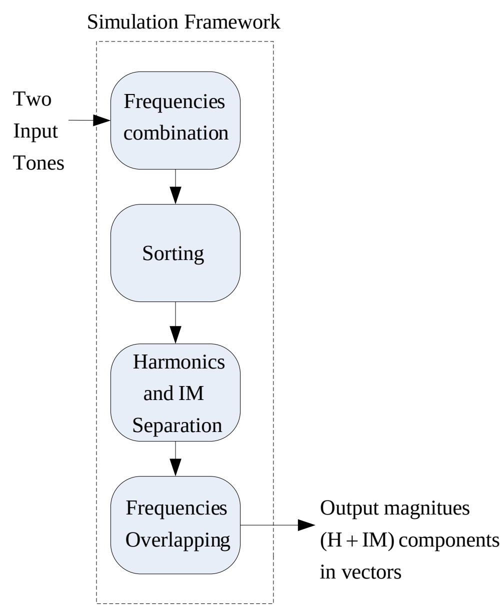

Figure 4: Simulation framework for multitones harmonic and intermodulation components analysis

The simulation framework in Figure 4 is for a single nonlinearity of order, n. As for the NLD case, where there are multiple order of different nonlinearities up to $n = 6$, weighted and connected in parallel, as shown in Figure 1. Therefore, we can link up the simulation framework for individual nonlinearity, add the vectors of harmonics and IM components and go through again *Sorting* and *Frequency Overlapping* stage for overall NLD. By doing this, we can get the harmonics and intermodulation components of polynomial approximated NLDs which are described in Section 4.

2.3. Harmonic Richness and Intermodulation Distortion Measurement Metrics

2.3.1. Total Harmonic Richness (THR)

In this paper, we define a term known as the total harmonic richness (THR) which is the ratio between the powers of NLD generated harmonics (from 1st to 6th order) to the power of the fundamental input tone. We include only the first six harmonics due to the findings from the region of dominance in psychoacoustic pitch perception researches [8-10, 17-19]. The *THR* formula for single tone analysis is given as

$$THR = \frac{(H_1)^2 + (H_2)^2 + \cdots + (H_6)^2}{(ST)^2}, \tag{16}$$

where the numerator is the summation of generated harmonics power from first to sixth harmonics, and the denominator is the power of the single tone which is the fundamental frequency.

2.3.2. Harmonic to IM Distortion Ratio ($HIDR$)

Harmonic to Intermodulation Distortion Ratio ($HIDR$) measures the ratio between the summation of the power of generated harmonics and the summation of the power of Intermodulation components. The *HIDR* formula for multitones analysis is

$$HIDR = \frac{\sum_{k=1}^{L}(H_k)^2}{\sum_{k=1}^{M}(IM_k)^2}, \tag{17}$$

where $\sum_{k=1}^{L}(H_k)^2$ is the summation of the power of L harmonic components, and $\sum_{k=1}^{M} IM_k^2$ is the summation of the power of M intermodulation distortion components, generated by the NLD or nonlinearity under investigation.

2.3.3. Harmonic to Multitones Ratio (Δ_H)

Harmonic to Multitones Ratio (Δ_H) measures the ratio between the power of generated harmonics and the summation of the power of input multitones. The Δ_H formula for multitones analysis is given as

$$\Delta_H = \frac{\sum_{k=1}^{L}(H_k)^2}{\sum_{i=1}^{N}(T_i)^2}, \tag{18}$$

where $\sum_{i=1}^{N}(T_i)^2$ is the summation of the power of input multitones and N is the number of multitones.

2.3.4. IM Distortion to Multitones Ratio (Δ_{IM})

Intermodulation Distortion to Multitones Ratio(Δ_{IM}) measures the ratio between the power of generated IM components and the summation of the power of input multitones. The Δ_{IM} formula for multitones analysis is given as

$$\Delta_{IM} = \frac{\sum_{k=1}^{M}(IM_k)^2}{\sum_{i=1}^{N}(T_i)^2}, \tag{19}$$

where $HIDR$, Δ_H and Δ_{IM} are objective measurement metrics for multitones input harmonic and intermodulation distortion analysis. *THR* metric is used for objective measurement index of the harmonic richness of a particular NLD under investigation. In this paper, these four metrics are used to compare the static memoryless NLDs of the virtual bass system.

3. NLD HARMONIC ANALYSIS

In this section, harmonic and intermodulation distortion analysis of five types of static nonlinear memoryless NLDs are presented. For each NLD, we presented the original system transfer functions, followed by polynomial approximated transfer functions. These transfer functions can be used to implement the NLD both in analog or digital systems.

To approximate the original system transfer function, we use MATLAB commands, such as *polyfit* and *polyval*. In this paper, half-wave rectifier, full-wave rectifier and limiter are approximated using these commands, and the resultant plots are presented. For the exponential function NLD, we can use Taylor's series approximate to obtain the polynomial form which is presented in the later section.

The main idea is to approximate the static memoryless NLDs using polynomials, reuse the polynomial harmonic and intermodulation analysis equations, and to study the effect of nonlinearity. We limit the maximum order to sixth order nonlinearity because of two reasons. The first reason is that according to psychoacoustic pitch perception researches findings [10][17-19], human are more sensitive up to sixth harmonics to create virtual bass effect. This is called dominance region in pitch perception researches. From the previous findings in Table 1, to generate up to sixth harmonics, we need up to sixth order nonlinearity. The seventh and more orders will generate the higher order harmonics which are not needed to create bass perception. The second reason is that the higher the order is, the lesser the contribution to harmonic generation and more intermodulation components are generated.

To obtain the amplitudes of DC and harmonics components from a NLD, we can make use of DFT or Schaefer-Suen equation (8) or directly applying the Fourier series to the single tone signal.

While using DFT, we have to select the frequency resolution, Δf which must be sharp enough to distinguish between two adjacent frequencies components. Another important factor to take into account is the maximum frequency, $f_{\max}$ it can capture by the DFT. These two relations can be described as follows:

$$\Delta f = \frac{f_s}{N}, \tag{20}$$

$$f_{\max} = \frac{f_s}{2}, \tag{21}$$

where f_s denotes the sampling frequency and N denotes the DFT points.

Since we are approximating all the NLDs, described in this paper as polynomial series, we can also make use of Schaefer-Suen equation (8) from Section 2.1. However, we limit the highest order up to six for the previously mentioned reasons. Therefore, the generated harmonics components can be computed easily using a calculator or MATLAB program. Therefore, using (8), the magnitude of the DC and harmonic components, $\{c_0, c_1, \cdots, c_6\}$ can be formulated as follows:

$$c_0 = 2 \times [h_0 + (\frac{A^2}{2})h_2 + (\frac{3A^4}{8})h_4 + (\frac{5A^6}{16})h_6], \tag{22}$$

$$c_1 = Ah_1 + (\frac{3A^3}{4})h_3 + (\frac{5A^5}{8})h_5, \tag{23}$$

$$c_2 = \frac{1}{2}[A^2 h_2 + A^4 h_4], \tag{24}$$

$$c_3 = \frac{1}{4}[A^3 h_3 + \frac{5A^5}{4} h_5], \tag{25}$$

$$c_4 = \frac{1}{8}[A^4 h_4 + \frac{3A^6}{4} h_6], \tag{26}$$

$$c_5 = \frac{1}{16}[A^5 h_5], \tag{27}$$

$$c_6 = \frac{3}{16}[A^6 h_6]. \tag{28}$$

3.1. Half-wave Rectifier NLD

The half-wave rectifier NLD system has a transfer function that can be expressed as

$$y = \frac{1}{2}(x + |x|). \tag{29}$$

Equation (29) can be approximated using polynomials as follows (up to 6 order)

$$y = 0.6535x^6 - 1.3296x^4 + 1.1390x^2 + 0.5x + 0.0419. \tag{30}$$

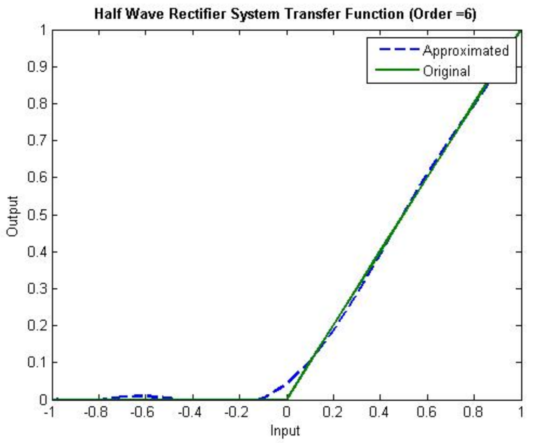

Figure 5: Half-wave Rectifier NLD original system transfer function and polynomial approximated transfer function.

Figure 5 shows the half-wave rectifier original and approximated system transfer functions. By using up to sixth order, the polynomials of (30) can approximate quite well to the original system function of (29). Figure 6 shows the NLD output in time domain using the original and approximated function. Figure 7 shows the frequency domain responses.

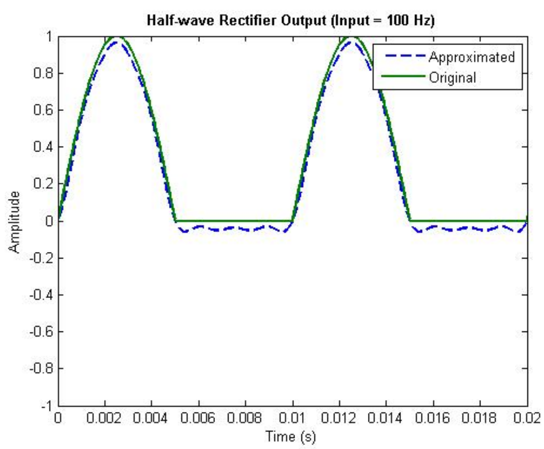

Figure 6: Half-wave Rectifier NLD single tone input responses for original transfer function and approximated transfer function.

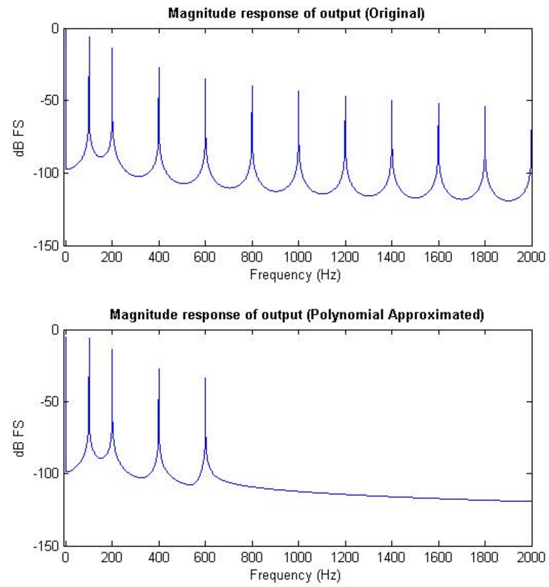

Figure 7: Half-wave Rectifier NLD single tone magnitude responses of original transfer function and approximated transfer function.

The upper plot of Figure 7 shows the harmonics, produced by the half-wave rectifier original system transfer function equation (29) and the lower plot shows the harmonics produced by the polynomial approximated equation (30). Half-wave rectifier produces DC, fundamental and even harmonics. The reason it can reproduce fundamental frequency can be linked to the polynomial approximated equation (30). In (30), due to the linear term, x, the fundamental frequency can be reproduced. Since we approximate up to the sixth order, the maximum harmonic number is six.

3.2. Full-wave Rectifier NLD

The full-wave rectifier NLD system transfer function can be expressed as

$$y = |x| . \qquad (31)$$

Equation (31) can be approximated using polynomials as

$$\mathrm{y} = 1.3070x^6 - 2.6593x^4 + 2.2781x^2 + 0.0838 . \qquad (32)$$

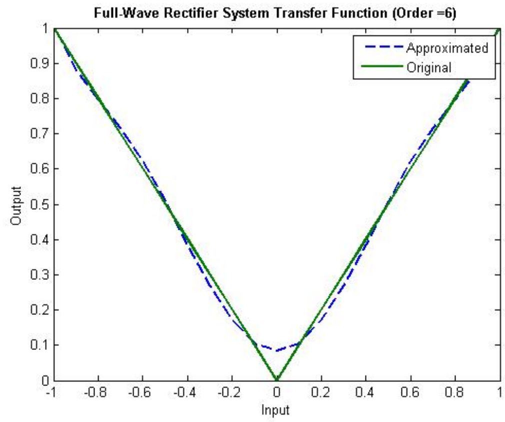

Figure 8: Full-wave Rectifier NLD original system transfer function and polynomial approximated transfer function.

Figure 8 shows the full-wave rectifier original system transfer function and polynomial approximated system transfer function. Figure 9 shows the single tone input response of full-wave rectifier NLD. From Figure 10, the original full-wave rectifier produces infinite series of harmonics, whereas the sixth order polynomial approximated function can produce up to sixth order harmonics, which are enough to create the virtual bass. Since the polynomial equation (32) has only even order nonlinearities, the generated harmonics are all even order harmonics, including DC component.

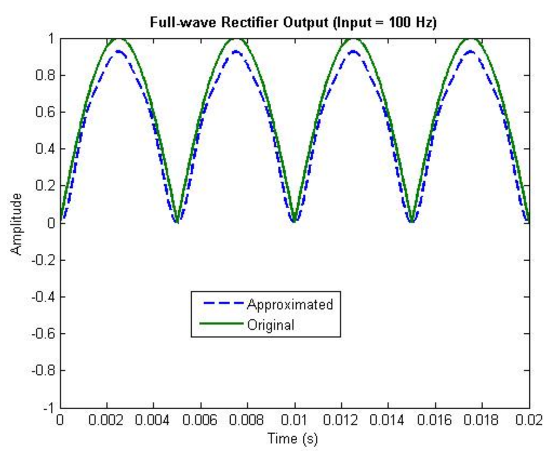

Figure 9: Full-wave Rectifier NLD single tone input responses for original transfer function and approximated transfer function.

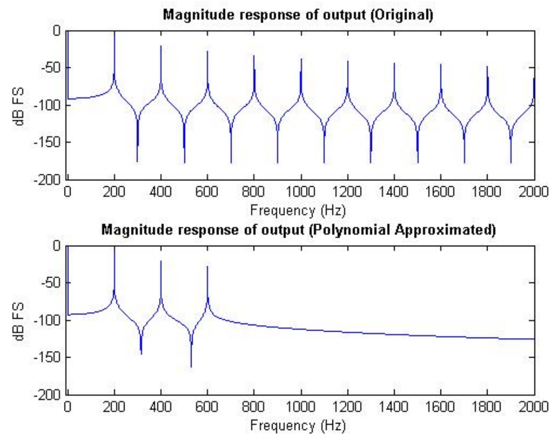

Figure 10: Full-wave Rectifier NLD single tone magnitude responses of original transfer function and approximated transfer function.

3.3. Square Wave Function NLD (Limiter)

The hard limiter or square wave function NLD transfer function can be mathematically described as

$$y = \begin{cases} +1, & x \geq 1 \\ 0, & x = 0 \\ -1, & x \leq 1 \end{cases} . \tag{33}$$

The transfer function can be approximated as

$$y = 4.4421x^5 - 7.2621x^3 + 3.9244x . \tag{34}$$

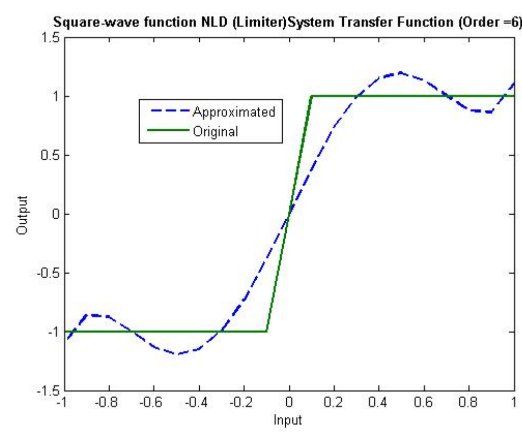

Figure 11: Square-wave function NLD original system transfer function and polynomial approximated transfer function.

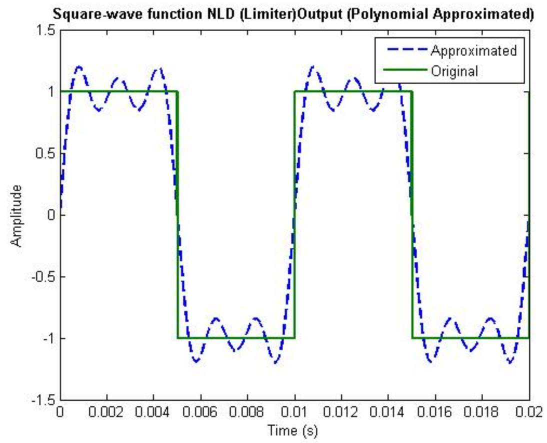

Figure 12: Square-wave function NLD single tone input responses for original transfer function and approximated transfer function.

Figure 11 shows the original and approximated transfer functions of limiter. In MATLAB simulation, we can use *sign* function for the original transfer function simulation. The polynomial approximated transfer function up to 6^{th} order is obtained as (34). The produced harmonics are all odd order harmonics, including fundamental component as shown in Figure 13.

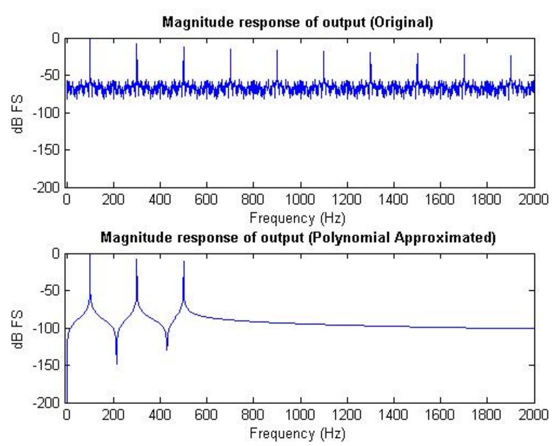

Figure 13: Square-wave function NLD single tone magnitude responses of original transfer function and approximated transfer function.

3.4. Exponential Function NLD

The exponential function can be expanded to a polynomial function by using Taylor's series. The expansion can be mathematically described as

$$y = f(x) = e^{x} = \sum_{k=0}^{\infty} \frac{(x)^{k}}{k!}. \tag{35}$$

Schaefer has previously derived the generalized single tone harmonic analysis equation for this exponential function based on his works on harmonic analysis for the bipolar junction transistor, and published his results in [16]. We modified his equation for the analysis of exponential function NLD used in the virtual bass system. The Schaefer's equation for the exponential function harmonic analysis can be expressed as

$$c_k = \left(\frac{A}{2}\right)^{k} \sum_{j=0}^{\infty} \frac{(A/2)^{2j}}{j!(k+j)!}. \tag{36}$$

Since $b^{x} = e^{x \ln b}$ and using (35), the exponential function with different base function can be converted to the polynomial form as

$$y = b^{x} = \sum_{k=0}^{\infty} \frac{(x \ln b)^{k}}{k!}. \tag{37}$$

To derive the harmonic analysis equation for (37), we modify the Schaefer's exponential function harmonic analysis equation (36) as

$$c_k = \left(\frac{A \ln b}{2}\right)^{k} \sum_{j=0}^{\infty} \frac{(A \ln b/2)^{2j}}{j!(k+j)!}. \tag{38}$$

Since we are approximating up to the sixth order, (38) can be expanded as

$$y = 1 + (\ln b)x + \frac{(\ln b)^2}{2}x^2 + \frac{(\ln b)^3}{6}x^3 + \frac{(\ln b)^4}{24}x^4$$

$$+ \frac{(\ln b)^5}{120}x^5 + \frac{(\ln b)^6}{720}x^6. \tag{39}$$

Therefore, we have two ways to perform harmonic analysis for exponential function NLD of base b. The first way is directly using (38) in infinite series form, and the second way is by using (39) that convert the exponential function into polynomial form up to 6th order and using the Schaefer-Suen equation of polynomial function harmonic analysis. By modifying the equation from exponential function of base e to the variable of base b, we have the exponential function base b as a parameter to control the harmonic richness and nonlinearity of the NLD. It is noted that the level of nonlinearity increases with a larger base constant, as plotted in Figure 14.

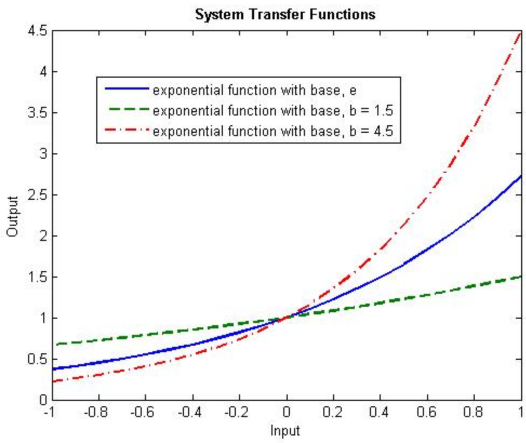

Figure 14: System transfer functions of exponential function with base e and adjustable base b.

For the case where the amplitude of the input signal is limited within $[-a, a]$, a modified exponential function can be used as follows:

$$y = \frac{b^x - b^{-a}}{b^a - b^{-a}}. \tag{40}$$

The modified and unmodified exponential functions are plotted in Figure 15 for case where $a = 1$ and $b = 1.5$. For input ranging from -1 to 1, the output of the modified exponential function is bounded within 0 to 1 as plotted in Figure 15. The exponential function NLD is computationally simple and it generates infinite series of odd and even order harmonics including the fundamental frequency and DC component. In Figure 16, the upper two plots are the original and polynomial approximated exponential functions with base, $b = 1.5$ and single tone input, $a = 1$. The lower two plots are the original and polynomial approximated exponential functions with base, $b = 1.5$ and single tone input, $a = 0.1$. From (39), the exponential function polynomial series has very large denominator terms for large order of polynomials. Therefore, the higher order terms are not significant at all.

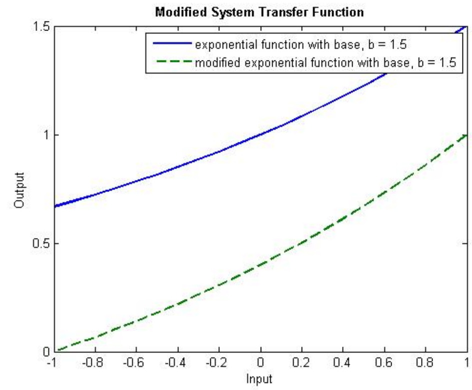

Figure 15: System transfer function of exponential function NLD of (40).

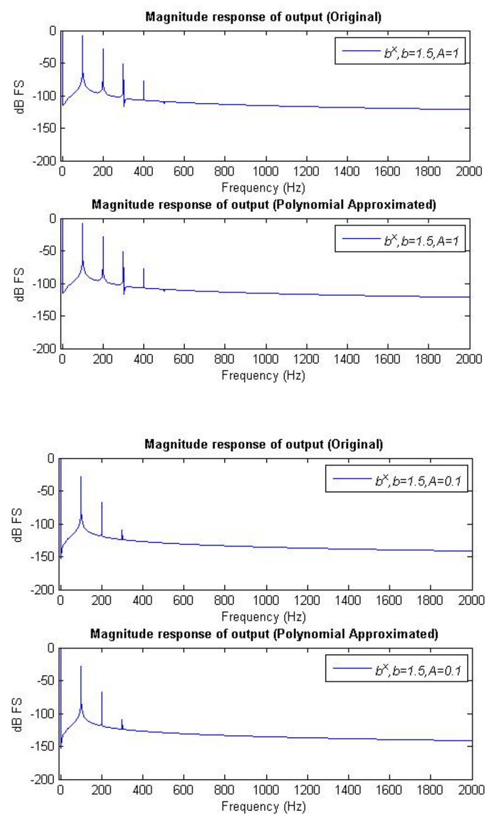

Figure 16: Exponential function NLD single tone magnitude responses of original transfer function and approximated transfer function.

3.5. Polynomial Function NLD

A polynomial function NLD can be constructed with weighted sum of nonlinearities, as shown in Figure 1. The even order and odd order nonlinearities, including $n = 1$ which is linear, are plotted in Figure 17. The even order nonlinearity can only produce even order nonlinearity due to its nature of nonsymmetrical nonlinearity, and odd order nonlinearity can only produce odd order nonlinearity due to the nature of symmetrical nonlinearity [25]. DC component is produced by even order nonlinearity. The maximum number of harmonics which can be produced is equal to the highest order of nonlinearity. The polynomial NLD generic equation up to sixth order nonlinearity can be described as

$$y = h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + h_5 x^5 + h_6 x^6 . \quad (41)$$

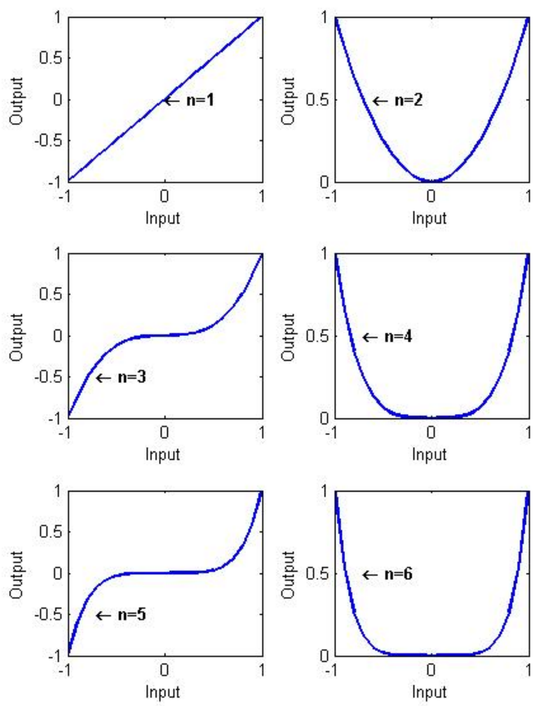

Figure 17: Linear and nonlinear system transfer function plots from $n = 1$ to $n = 6$, where n is the order of nonlinearity.

The polynomial function NLD is generic and as shown in previous sections, the first four NLD can be approximated as polynomial function NLD. The design parameters are weights such as $\{h_1, h_2, \cdots, h_6\}$, and adding or removing nonlinearities. As described in Section 2, the Schaefer-Suen equation (8) can be applied to perform single tone harmonic analysis and numerical analysis framework of Figure 4 can be used to carry out multitones intermodulation distortion analysis. The next section presents the single tone and logarithmic multitones harmonic and intermodulation distortion analysis results.

4. RESULTS AND DISCUSSIONS

In this section, the single tone and logarithmic multitones analysis results of four different NLDs are presented. Polynomial NLD is omitted because of its generic nature, wide design freedom (parameters), and all other NLDs can be approximated using polynomial series, forming a kind of polynomial NLD. Four measuring metrics, such as THR, $HIDR$, Δ_H and Δ_{IM}, are used to compare among the four different NLDs. All the respective equations are described in (16), (17), (18), and (19) of Section 2.3. We use THR to compare the single tone analysis results, and $HIDR, \Delta_H$ and Δ_{IM} to compare the harmonic and intermodulation distortion results among the four NLDs. The four measurement metrics described in Section 2.3 can be simply be summarized in word as follows:

$$THR = \frac{\sum \text{Power of the first six harmonics}}{\text{Power of input Single Tone}} , \quad (42)$$

$$HIDR = \frac{\sum \text{Power of generated harmonics}}{\sum \text{Power of generated IM components}} , \quad (43)$$

$$\Delta_H = \frac{\sum \text{Power of generated harmonics}}{\text{Power of input Multitone}} , \quad (44)$$

$$\Delta_{IM} = \frac{\sum \text{Power of generated IM components}}{\text{Power of input Multitone}} . \quad (45)$$

THR metric of (16) or (42) shows how much richer the power of harmonics are, based on the single tone input to the NLD under investigation.

If only the single tone is fed into the NLD, there is no intermodulation component at the output. To excite as many intermodulation distortion components as possible at the output, we used five logarithmic multitones (calculated in Section 2.2, Table 2.) as input to the NLD under investigation. The $HIDR$ metric of (17) or (43) give us a measurement of the power ratio between the harmonics and intermodulation distortion components, based on the logarithmic multitones input. Δ_H metric of (18) or (44) represents the power ratio between the generated harmonics and power of five logarithmic multitones inputs. It only takes into consideration of harmonics components, generated by the multitones stimulus. On contrary, Δ_{IM} of (19) or (45) represents the power ratio between the generated intermodulation distortion components only and power of five logarithmic multitones inputs. In this case, it only takes into account of intermodulation distortion components, generated by the multitones stimulus. By using Δ_H and Δ_{IM} , we can decouple the harmonics and intermodulation components at the output and study their effects. We vary the amplitude of the single tone input (A) from 0.1 to 1, calculate the metric and plot the responses in next sections. In the case of exponential NLD, we can also vary the b parameter and study its effects.

4.1. Single Tone Harmonic Analysis Results

In this section, THR performance of four different NLDs are presented by varying the amplitude of the input tone, A . As for the exponential function, NLD, we vary the base, b and the amplitude of the input tones, A .

Figure 18 shows the THR plot of half-wave and full-wave rectifier NLDs. Based on this plot, half-wave rectifier can produce richer power of harmonics than the full-wave rectifier does. The THR difference is 0.1 between the two NLDs. The varying input amplitude of single tone does not affect the THR performance for both rectifiers. We only take into consideration of the power of harmonic components. As mentioned in the previous section, the half-wave rectifier can only produce even order harmonics, including DC component and fundamental. Except the fundamental component, it cannot generate the odd order harmonics. The full-wave rectifier generates the DC, and only even order harmonics. It cannot generate the odd order harmonics and the fundamental component. That is the reason why THR values of full-wave rectifier is always less than those of half-wave rectifier by 10% for all input range from 0.1 to 1.

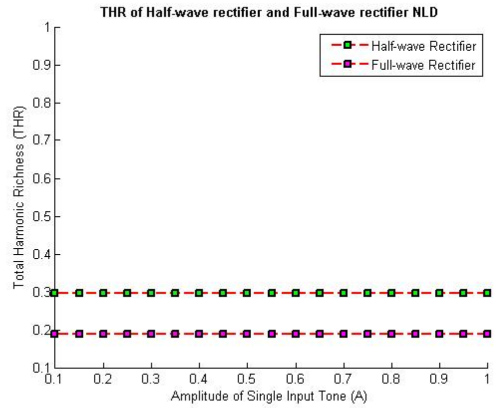

Figure 18: Total Harmonic Richness (THR) plot of Half-wave and Full-wave Rectifier with single tone input amplitude (A) .

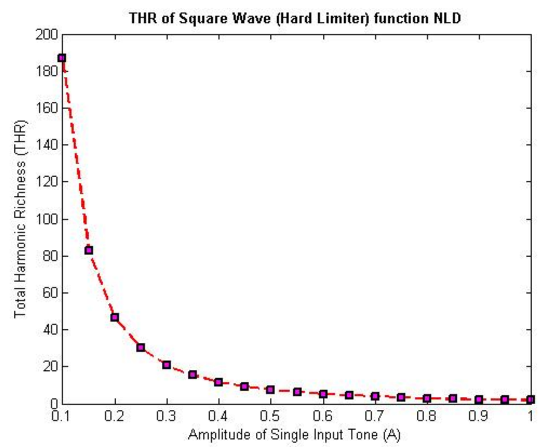

Figure 19: Total Harmonic Richness (THR) plot of Square Wave Function NLD with single tone input amplitude (A) .

Figure 19 shows the THR plot of square wave function NLD or a hard limiter. Referring to (34), the square wave function NLD can be approximated by only odd order polynomials or odd order nonlinearities. From the findings in Table 1 (Section 2.1), odd order nonlinearities can never produce DC component, but it always reproduce the fundamental component which is the first harmonic. From (34), the polynomial equation has a linear term, x multiplied by 3.9244. The linear term or linearity always produces the fundamental term, and in this case, it is weighed by 3.9244. Since the square wave function NLD is a nonlinear system constructed with all odd order nonlinearities, it does not generate DC components. Therefore, all the energies are concentrated at only odd order harmonics with very strong energy at the first harmonic, or reproduced fundamental frequency. The sum of the power of the harmonics (1^{st}, 3^{rd}, 5^{th} } is always the same no matter how much is the input power of amplitude of the single tone, A. That is the reason why the small amplitude, for instant, 0.1 has a very high THR value of 186.6, and the higher amplitude, 1, has a low THR value of 1.866.

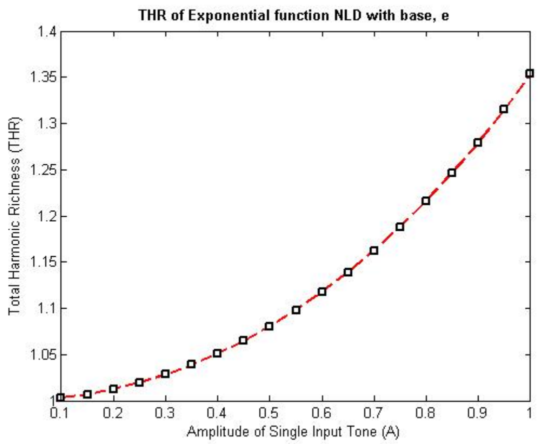

Figure 20: Total Harmonic Richness (THR) plot of Exponential Function, e^x NLD with single tone input amplitude (A).

Figure 20 shows the THR plot of the exponential function NLD, e^x. Note that e^x is a special case of b^x with base, $b = e.$ As described in (35) and (37), the exponential function consists of a linear term and both even and odd order nonlinear terms (nonlinearities). Therefore, this NLD generates DC, fundamental frequency and all even and odd order harmonics. Deduced from Schaefer's equations of (36) and (38), the input signal amplitude has significant influence on the generated harmonics' power. In addition, the NLD generates higher power of harmonics when the input signal amplitude is high, whereas it generates very low harmonics' power when the input signal level is low. The harmonics decay rate is also very high for the exponential function NLD, as plotted in Figure 16 with comparison of $A = 0.1$ and $A = 1.0$. Hence, the higher order harmonics' powers are not significant at all when the input signal amplitude is very low. Figure 20 shows the same interpretation with higher input signal amplitude generates richer harmonic power. Unlike previous three NLDs, such as half-wave rectifier, full-wave rectifier and hard limiter, the exponential NLD generates both even and odd harmonics.

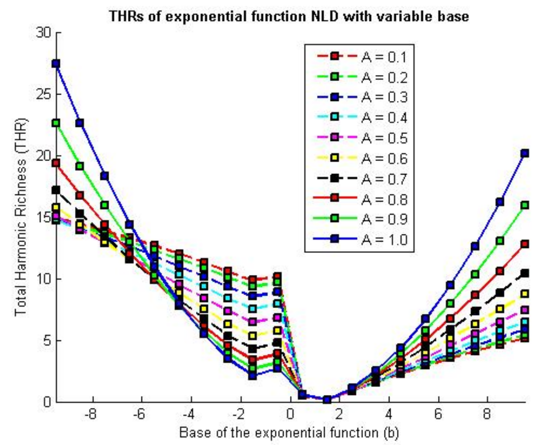

Figure 21: Total Harmonic Richness (THR) plot of Exponential Function, b^x NLD with single tone input amplitude (A), and base (b) as varying parameters.

Figure 21 is an additional simulation results for the exponential function, b^x with adjustable base, b as a parameter. We vary the base, b from -9.5 to +9.5 with a step size of 0.5. This plot also includes the input signal amplitude, A as a varying parameter. Figure 20 shows that the higher THR values are generated by the higher input signal level. This is for the case of e^x with fixed base, e. As for the case of b^x with base as a parameter, it depicts a new finding such that the higher input level generates higher THR values only when the base, b is positive. The negative base, b has opposite effect, meaning that the lower input signal generates higher THR from $b = 0$ to $b = -6.5$. However, for the case of $b < -6.5$, it generates higher THR for higher input signal level.

4.2. Multitones Harmonic and Intermodulation Distortion Analysis Results

MATLAB simulations are carried out to study the intermodulation effects of NLD, fed by multitones stimulus. The five logarithmic multitones are calculated using Table 2. The polynomial approximation equations for five different NLDs are described in Section 3. For every nonlinearity in the approximation equations, the frame work of Figure 4 is used to calculate the total number of harmonics and intermodulation distortion components and their respective frequency values. The framework of Figure 22 combines all the harmonics and intermodulation components vector, sorts again and overlap the frequencies' indexes and generate two vectors, showing which frequencies are harmonics and intermodulation distortion components. DFT was performed for the signals output of original NLD transfer functions. Input signal sampling frequency is set as 48,000 Hz. Therefore, the DFT frequency resolution is 1 Hz, using (20). Since the generated harmonics and intermodulation components are integer values and FFT bin resolution is 1 Hz, we can directly set the generated frequencies to the DFT bin indexes in simulation. The highest order of nonlinearity used in simulation is the sixth order. The maximum frequency of the multitones is 200 Hz (Table 2). Therefore, the maximum frequency the system can generate is 1200 Hz which is less than Nyquist Frequency 24,000 Hz. The sampling frequency fold-over or aliasing effect will not occur in the simulation.

Figures 23 to 26 present the simulation results for four different NLDs under investigation. *HIDR* values are constant across varying input amplitudes from 0.1 to 1 in all figures. This means that the power ratio between harmonic and intermodulation distortion is the same no matter what is the level of the input tones. Δ_{IM} values are large when the input multitones amplitudes are small. This can be interpreted as at low input signal level, all four NLDs generated more IM components' powers with respect to the input multitones power. For the full-wave and half-wave rectifiers, the drop in Δ_{IM} power ratio is not that significant, compared to the drastic power ratio drop of exponential function NLD and square-wave NLD at higher input levels. Therefore, this result can be interpreted as when the input signals level is low, the exponential function NLD and the square-wave NLD generate stronger intermodulation distortion components' powers, whereas the rectifiers NLD, even though they generate stronger intermodulation component power, the ratio is not significant. They all agree that at the higher input signal levels, the powers of generated intermodulation components are lower. Another measure, which is used to calculate the power ratio between the total harmonic power and the input multitones power, is Δ_H . All four NLDs have less varying Δ_H across the varying input signal amplitudes. The general trend of Δ_H is approaching zero when the input signal level is higher. This means that at the higher input signal level, the generated harmonics power are almost negligible with respect to the input signal power. In general, at the higher input signal level, the power of intermodulation components and harmonics, generated by multitones inputs are lesser.

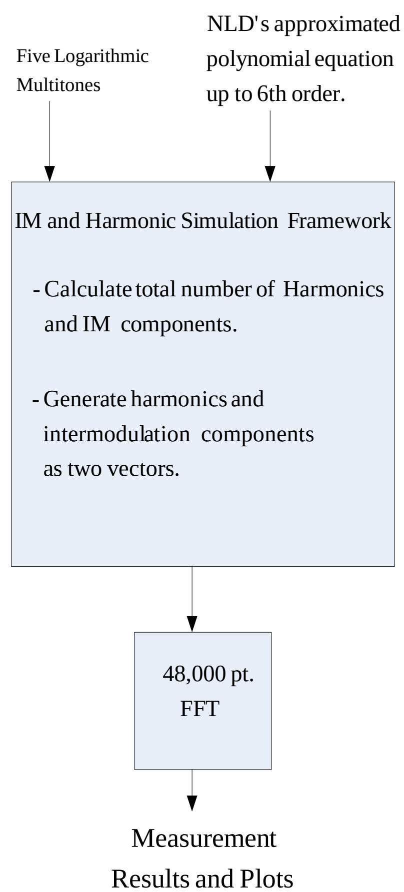

Figure 22: Harmonic and Intermodulation (IM) Analysis Simulation Block Diagram.

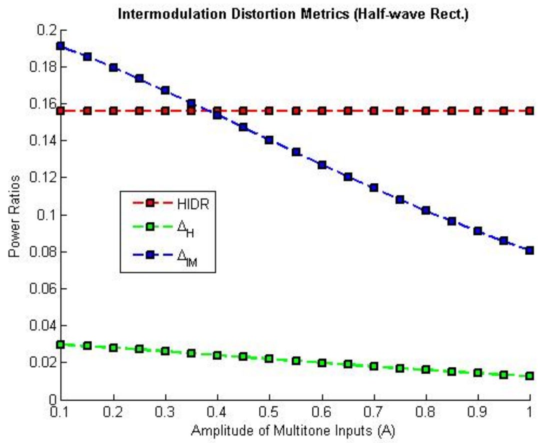

Figure 23: $HIDR, \Delta_H$ and Δ_{IM} intermodulation distortion ratio metrics simulation plots of Half-wave rectifier NLD with multitones inputs (A).

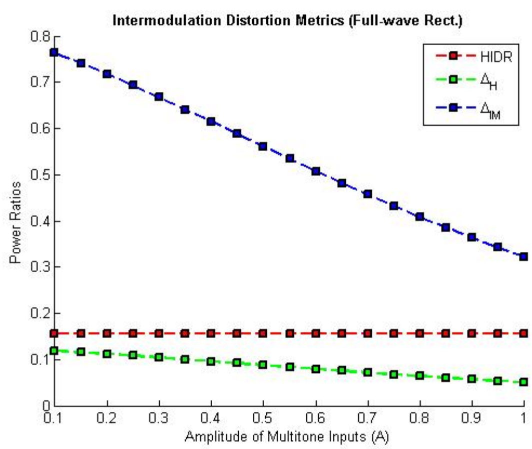

Figure 24: $HIDR, \Delta_H$ and Δ_{IM} intermodulation distortion ratio metrics simulation plots of Full-wave rectifier NLD with multitones inputs (A).

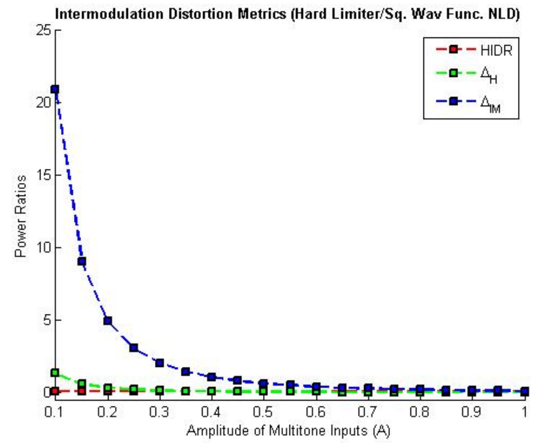

Figure 25: $HIDR, \Delta_H$ and Δ_{IM} intermodulation distortion ratio metrics simulation plots of Square-wave function NLD with multitones inputs (A).

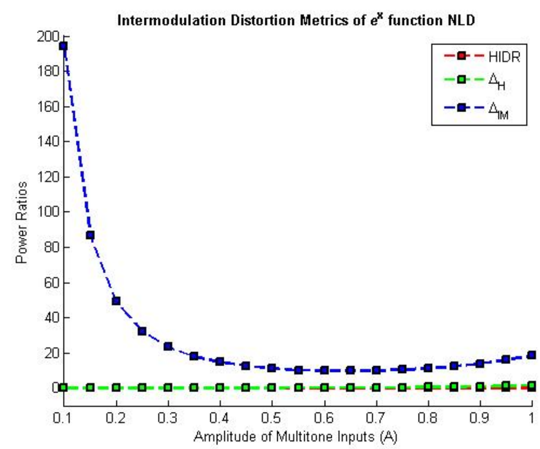

Figure 26: $HIDR, \Delta_H$ and Δ_{IM} intermodulation distortion ratio metrics simulation plots of Exponential function, e^x NLD with multitones inputs (A).

5. CONCLUSION

Harmonic analysis of single tone and logarithmic multitones for the NLDs in virtual bass system was presented in this paper. The original system transfer functions of static memory-less NLDs are approximated using polynomial series up to 6^{th} order. Mathematical harmonic analysis formulae were used to develop the simulation framework. This simulation framework was then used to obtain the THR, $HIDR$, Δ_H and Δ_{IM} values for single tone and multitones NLD harmonic analysis. Results were plotted, interpreted and related with psychoacoustic pitch perception researches to study the performance of full-wave rectifier, half-wave rectifier, hard limiter and exponential function NLDs.

The half-wave rectifier generates the fundamental and even harmonics only, whereas the full-wave rectifier generates only even harmonics. Then the perceived pitch is one octave higher than the original pitch [1]. The varying input signal level has no significant effect upon THR values of half- and full-wave rectifiers, as shown in Figure 18. This means that no matter what the input power is, the generated harmonics' powers are the same.

The hard limiter or square-wave function NLD generates only odd harmonics without DC and fundamental component. Just like half- and full-wave rectifiers, the hard limiter NLD produces an octave high pitch perception, which is not matched to the original pitch. From Figure 19, the THR values are very high when the input signal level is low, but the THR values are very low when the input signal level is high. The general requirement of the virtual bass system is that when the input is low, the generated harmonic power should be low, and vice versa. The harmonic generation effect, based on THR plot (Figure 19) has the reversed effect which is not desired.

The harmonic components generated by the exponential NLD have both even and odd ordered components, including fundamental component. From Figure 16, when the input signal level is low, the generated harmonics' levels are low. When the input signal level is high, the generated harmonics levels are high. Figure 20 presents the same characteristic, which is desired to create virtual pitch which is matched to the original pitch of the signal. In addition, the exponential function NLD has a parameter, base *b*, to control the harmonic richness. Based on these simulation results, exponential NLD is considered the best for virtual bass systems.

From Figures 23 to 26, the power of intermodulation components are very strong when the input signal level is very low, which is depicted by the Δ_{IM} plot for all four NLDs. But, it decreases rapidly when the input signal level goes higher. Therefore, it is recommended not to feed the very low input signal into the NLD devices.

6. ACKNOWLEDGEMENTS

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